1 D Regularisation

1. Coordinate Transformation $r = u^2$

$$H = \frac{p^2}{2\mu} - \frac{GMm}{r} = E_0 = \text{const.} \quad p = \mu\dot{r} = 2u\dot{u}\mu$$

Canonical Trafo: $p\dot{r} = P\dot{u} \Rightarrow P = 4u^2\dot{u}\mu$

$$H = \frac{P^2}{8u^2\mu} - \frac{GMm}{u^2} = \text{const. } E_0$$

2. Time Transformation

$$dt = r\,ds = u^2 ds \; ; \quad \dot{u} = \frac{du}{dt} = \frac{1}{r}\frac{du}{ds} \Rightarrow$$

$= g(p,r)$
$= g(P,u)$

$$u^2\dot{u} = u' = \frac{1}{4}\frac{P}{\mu}$$

3. Poincare - Transform:

$$0 = \Gamma = g(P,u)\,H(P,u) = \frac{P^2}{8\mu} - GMm - E_0 u^2$$

4. Canonical Eq: $P' = \frac{\partial\Gamma}{\partial u} = -2E_0 u = 4u''\mu$

$$\Rightarrow u'' + \frac{1}{2}\frac{E_0}{\mu}u = 0$$ harmonic oscillator (if $E_0 < 0$)

$$\omega^2 = \frac{E_0}{2\mu}$$ half frequency

2D

Need

$$r = L(u)u = u \circ u = u^2 = \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]^2$$

$$\mathring{r} \doteq 2 u \mathring{u}$$

Vector!

Complex numbers

Levi-Civita

Levi-Civita Transformation 2-D

$$H = \mu \frac{V^2}{2} - \frac{Gm_1 m_2}{R} = H(V, R)$$

$$(u \cdot u)^2 = R \iff L(u)u = R \iff L(u) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$$

$$2L(u)\dot{u} = \dot{R} = V$$

$$4u^2 \dot{u}^2 = \dot{R}^2 = V^2 .$$

$$\Rightarrow \quad P = 4u^2 \dot{u} = 2L^T(u) \cdot V$$

$$\Rightarrow V^2 = \frac{P^2}{4u^2}$$

$$\Rightarrow \quad \frac{1}{2u^2} L(u) \cdot P = V$$

$$H = \mu \frac{P^2}{8u^2} - \frac{Gm_1 m_2}{u^2} = H(P, u)$$

Time Transformation

$$\frac{dt}{ds} = R = u^2$$

Poincaré Transform

$$\Gamma = R(H - E) = \frac{\mu P^2}{8} - Eu^2 - Gm_1 m_2$$

harmonic oscillator $\omega^2 = \frac{\mu}{4E}$

half frequency

Are our transformations canonical?
(Levi-Civita plus time transformation)

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$ S: action integral
L: Lagrange function

$\delta S = 0$ defines physical motion
(least action principle, Hamilton's variational princ.)

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$ = (partial integr.)
(variat. of boundary terms vanishes)

$$= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt$$

For all $\delta q_i \Rightarrow$ Euler-Lagrange Eq. of motion!

$$H = \sum_i p_i \dot{q}_i - L \iff L = \sum_i p_i \dot{q}_i - H$$

$$0 = \delta S = \int_{t_1}^{t_2} \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right) dt$$

$$= \int_{t_1}^{t_2} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] dt$$

For all $\delta q_i, \delta p_i \Rightarrow$ Hamilton's Eq. of motion.

Canonical Transformation $\Longleftrightarrow$

$\Longleftrightarrow$ Hamilton's Eq. of motion invariant

$$\delta \int_{t_1}^{t_2} \left(\sum p_i \dot{q}_i - H \right) dt = 0$$ in both systems of variables.

1) Levi-Civita transformation

$$\sum_{i=1}^{3} p_i \dot{q}_i = \sum_{i=1}^{3} v_i \dot{r}_i = \sum_{i=1}^{3} v_i^2 \qquad \begin{matrix} p_i = v_i \\ q_i = r_i \end{matrix}$$

Let $Q_i = u_i$, $P_i = 4u^2 \dot{u}_i$ Levi-Civita variables

$$\sum_{i=1}^{4} P_i \dot{Q}_i = 4u^2 \sum_{i=1}^{4} \dot{u}_i^2 = \sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{3} p_i \dot{q}_i$$

since $\sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{4} 4u^2 \dot{u}_i^2$ see before

---

2) Time transformation $dt = g(Q, t)\, ds$

$$\delta S = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{3} p_i \dot{q}_i - H(p,q) \right) dt = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{4} P_i \dot{Q}_i - H(P,Q) \right) dt$$

$$= \delta \int_{s_1}^{s_2} \left(\frac{1}{g} \sum_{i=1}^{4} P_i Q_i' - H(P,Q) \right) g\, ds$$

$$\boxed{Q_i' = \frac{\partial Q_i}{\partial s} = \frac{\partial Q_i}{\partial t} \cdot \frac{\partial t}{\partial s} = \dot{Q}_i \cdot g \, ; \, \Gamma = g\left(H - \frac{h}{\mu}\right)} = \delta \int_{s_1}^{s_2} \left(\sum_{i=1}^{4} P_i Q_i' - \Gamma \right) ds$$

③

$p_i, q_i, H(p_i, q_i, t), t$ satisfies Hamilton's eq.

$\Longleftrightarrow$

$P_i, Q_i, \Gamma(P_i, Q_i, s), s$ satisfies Hamilton's eq.

Γ: Poincaré transform of Hamiltonian

$\Gamma = g\left(H - \frac{h}{\mu}\right) \qquad dt = g \cdot ds$

Our example: $\qquad g = r = u^2 = \sum_{i=1}^{4} u_i^2$

$$H = \frac{1}{2}\sum_{i=1}^{3} v_i^2 - \frac{Gm_1m_2}{\mu r} = 2u^2 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu u^2}$$

$$\Gamma = r\cdot\left(H - \frac{h}{\mu}\right) = 2u^4 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu} - \frac{h}{\mu}u^2$$

$$= \sum_{i=1}^{4}\left(\frac{P_i^2}{8} - \frac{h}{\mu}Q_i^2\right) - \frac{Gm_1m_2}{\mu}$$

Canonical Equations:

$$Q_i' = \frac{\partial \Gamma}{\partial P_i} = \frac{P_i}{4} \Longleftrightarrow u_i' = u^2\dot{u}_i = r\dot{u}_i \quad \text{O.K.}$$

$$P_i' = -\frac{\partial \Gamma}{\partial Q_i} = 2\frac{h}{\mu}Q_i \Longleftrightarrow \left(4u^2\dot{u}_i\right)' = 4u_i'' = 2\frac{h}{\mu}u_i$$

$u_i'' - \frac{h}{2\mu}u_i = 0$	$u_i'' + \frac{\lvert h\rvert}{2\mu}u_i = 0 \quad (h<0)$

Henceforth:

Time transformation $dt = R_1 R_2 \, ds$

Poincaré Hamiltonian $\Gamma = R_1 R_2 \left(H - \quad\right)$

e.g. $\vec{R}_1 = \vec{r}_1 - \vec{r}_3$ for three bodies!
$\vec{R}_2 = \vec{r}_2 - \vec{r}_3$

Hamiltonian equations $\Longrightarrow$

regularized equations of motion.

Next: Aarseth + Zare '74:
3-body regularization.

16.6.99 (1)

Three-Body Regularization

i) Three-Body Hamiltonian

$$H = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2\right)$$

$$- \frac{G m_1 m_2}{r_1 - r_2} - \frac{G m_2 m_3}{r_2 - r_3} - \frac{G m_1 m_3}{r_1 - r_3}$$

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3} = \frac{\sum m_i \vec{v}_i}{M}$$

$$\vec{V}_1 = \vec{v}_1 - \vec{v}_3 \qquad \vec{R}_1 = \vec{r}_1 - \vec{r}_3$$

$$\vec{V}_2 = \vec{v}_2 - \vec{v}_3 \qquad \vec{R}_2 = \vec{r}_2 - \vec{r}_3$$

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \qquad r_i = |\vec{r}_i| \quad v_i = |\vec{v}_i|$$

$$R_i = |\vec{R}_i| \quad V_i = |\vec{V}_i|$$

2

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 \quad (1)$$

$$m_3\vec{V}_1 = m_3\vec{v}_1 - m_3\vec{v}_3 \quad (2)$$

$$m_3\vec{V}_2 = m_3\vec{v}_2 - m_3\vec{v}_3 \quad (3)$$

$$M\vec{v}_{cm} + m_3\vec{V}_1 = (m_1+m_3)\vec{v}_1 + m_2\vec{v}_2 \quad (4)=(1)+(2)$$

$$M\vec{v}_{cm} + m_3\vec{V}_2 = (m_2+m_3)\vec{v}_2 + m_1\vec{v}_1 \quad (5)=(1)+(3)$$

$$(m_2+m_3)M\vec{v}_{cm} + (m_2+m_3)m_3\vec{V}_1 = (m_2+m_3)(m_1+m_3)\vec{V}_1 + (m_2+m_3)m_2\vec{v}_2$$

$(4) * (m_2+m_3) = (6)$

$$-m_2M\vec{v}_{cm} - m_2m_3\vec{V}_2 = -m_1m_2\vec{v}_1 - m_2(m_2+m_3)\vec{v}_2$$

$(5) * (-m_2) = (7)$

$$m_3M\vec{v}_{cm} + (m_2+m_3)m_3\vec{V}_1 - m_2m_3\vec{V}_2 =$$

$$(8) \qquad = (m_2+m_3)(m_1+m_3)\vec{v}_1 - m_1m_2\vec{v}_1$$

$$= (m_2m_3 + m_3(m_1+m_3))\vec{v}_1 = m_3M\vec{v}_1$$

$$\vec{v}_1 = \vec{v}_{cm} + \frac{m_2+m_3}{M}\vec{V}_1 - \frac{m_2}{M}\vec{V}_2 \qquad (8)/(m_3M)$$

$$\vec{v}_2 = \vec{v}_{cm} + \frac{m_1+m_3}{M}\vec{V}_2 - \frac{m_1}{M}\vec{V}_1 \qquad (9)$$ analogous 1 and 2 exchanged

$$\vec{V}_3 = \left(M\vec{v}_{cm} - m_1\vec{v}_1 - m_2\vec{v}_2 \right) / m_3 =$$

$$= \Big(M\vec{v}_{cm} - m_1\vec{v}_{cm} - \frac{m_1}{M}(m_2+m_3)\vec{V}_1 + \frac{m_1 m_2}{M}\vec{V}_2$$
$$- m_2\vec{v}_{cm} - \frac{m_2}{M}(m_1+m_3)\vec{V}_2 + \frac{m_1 m_2}{M}\vec{V}_1 \Big) / m_3$$

$$= \vec{v}_{cm} - \frac{m_1}{M}\vec{V}_1 - \frac{m_2}{M}\vec{V}_2 \qquad (10)$$

Transformation:

$$T = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2\right) = \frac{1}{2}\Big(M v_{cm}^2 + m_1\frac{(m_2+m_3)^2}{M^2}V_1^2 + \frac{m_1 m_2^2}{M^2}V_2^2$$
$$+ m_2\frac{(m_1+m_3)^2}{M^2}V_2^2 + \frac{m_2 m_1^2}{M^2}V_1^2 + \frac{m_3 m_1^2}{M^2}V_1^2 + \frac{m_3 m_2^2}{M^2}V_2^2$$
$$- 2\frac{m_1 m_2(m_2+m_3)}{M^2}V_1V_2 - 2\frac{m_1 m_2(m_1+m_3)}{M^2}V_1V_2 + 2\frac{m_1 m_2 m_3}{M^2}V_1V_2$$
$$+ 2\frac{m_1(m_2+m_3)}{M}V_1 v_{cm} + 2\frac{m_2(m_1+m_3)}{M}V_2 v_{cm} - 2\frac{m_1 m_3}{M}V_1 v_{cm}$$
$$- 2\frac{m_1 m_2}{M}V_2 v_{cm} - 2\frac{m_1 m_2}{M}V_1 v_{cm} - 2\frac{m_2 m_3}{M}V_2 v_{cm}\Big)$$

Collect Terms: $V_1 V_2$:

$$\frac{2}{M^2}\left(m_1 m_2 m_3 - m_1 m_2(m_1+m_3) - m_1 m_2(m_2+m_3)\right) =$$
$$= \frac{2}{M^2}\left(-m_1 m_2 m_3 - m_1^2 m_2 - m_1 m_2^2\right) = -\frac{2 m_1 m_2}{M}$$

Collect Terms V_1^2:

$$\frac{1}{M^2}\left(m_1 (m_2+m_3)^2 + m_2 m_1^2 + m_3 m_1^2 \right) =$$

$$= \frac{1}{M^2}\left(m_1 m_2^2 + 2 m_1 m_2 m_3 + m_1 m_3^2 + m_2 m_1^2 + m_3 m_1^2 \right)$$

$$= \frac{1}{M^2} m_1 \left(m_2^2 + m_3^2 + 2 m_2 m_3 + m_1 (m_2+m_3) \right)$$

$$= \frac{1}{M^2} m_1 (m_2+m_3)(m_1+m_2+m_3) = \frac{m_1 (m_2+m_3)}{M}$$

Analogous V_2^2: $\dfrac{m_2 (m_1+m_3)}{M}$

All Terms $V_1 v_{cm}$, $V_2 v_{cm}$ vanish! $\Longrightarrow$

$$H = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 (m_2+m_3)}{M} V_1^2 + \frac{1}{2} \frac{m_2 (m_1+m_3)}{M} V_2^2$$

$$- \frac{m_1 m_2}{M} V_1 V_2 - \frac{G m_1 m_2}{R_1 - R_2} - \frac{G m_2 m_3}{R_2} - \frac{G m_1 m_3}{R_1}$$

Note: $r_1 - r_2 = R_1 - R_2$ has been used.

Note: $V_1 - V_2 = V_{rel}$, $R_1 - R_2 = R_{rel}$ $\Longrightarrow$

$$H = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{M} V_{rel}^2 - \frac{G m_1 m_2}{R_{rel}}$$

$$+ \frac{1}{2} \frac{m_1 m_3}{M} \left(V_1^2 + \frac{m_2 m_3}{M} V_2^2 \right) - \frac{G m_2 m_3}{R_2} - \frac{G m_1 m_3}{R_1}$$

Regularization:

$\vec{Q}_i, \vec{P}_i \in \mathbb{R}^4$

$dt = R_1 R_2\, ds$

$R_1 = Q_1^2 \qquad \vec{R}_1 = L(\vec{Q}_1)\vec{Q}_1$

$R_2 = Q_2^2 \qquad \vec{R}_2 = L(\vec{Q}_2)\vec{Q}_2$

$\vec{P}_1 = 4Q_1^2 \dot{\vec{Q}}_1 = 4\vec{Q}_1' / Q_2^2 \quad \vec{V}_1 = 2 \cdot L(\vec{Q}_1) \quad \frac{\vec{P}_1}{4Q_1^2}$

$\vec{P}_2 = 4Q_2^2 \dot{\vec{Q}}_2 = 4\vec{Q}_2' / Q_1^2 \quad \vec{V}_2 = 2 \cdot L(\vec{Q}_2) \quad \frac{\vec{P}_2}{4Q_2^2}$

$$V_1^2 = \dot{R}_1^2 = 4Q_1^2 \dot{Q}_1^2 = \frac{P_1^2}{4Q_1^2}; \quad V_2^2 = \frac{P_2^2}{4Q_2^2}$$

$$\Gamma = R_1 R_2 (H - E) = Q_1^2 Q_2^2 (H - E)$$

$$= \frac{m_1(m_2+m_3)}{M} \frac{P_1^2 Q_2^2}{8} + \frac{m_2(m_1+m_3)}{M} \frac{P_2^2 Q_1^2}{8}$$

$$- \frac{m_1 m_2}{M} \cdot \frac{L(\vec{Q}_1)\vec{P}_1 \, L(\vec{Q}_2)\vec{P}_2}{4}$$

$$- \frac{G m_1 m_2 Q_1^2 Q_2^2}{|\vec{R}_1 - \vec{R}_2|} \; G m_2 m_3 Q_1^2 - G m_1 m_3 Q_2^2 - E Q_1^2 Q_2^2$$

Next: Equations of Motion!

3-6 Equations of Motion

30.6.99 (1)

$$\vec{Q}_1' = \frac{\partial \Gamma}{\partial \vec{P}_1} = \frac{m_1(m_2+m_3)}{M}\frac{P_1 Q_2^2}{4} - \frac{m_1 m_2}{M}\frac{L^T(\vec{Q}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$\vec{P}_1' = -\frac{\partial \Gamma}{\partial \vec{Q}_1} = \frac{m_2(m_1+m_3)}{M}\frac{\vec{Q}_1 P_2^2}{4} - \frac{m_1 m_2}{M}\frac{L^T(\vec{P}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$- \frac{2 G m_1 m_2}{|R_1 - R_2|}\vec{Q}_1 Q_2^2 - 2 G m_2 m_3 \vec{Q}_1 - 2E\vec{Q}_1 Q_2^2$$

$$\vec{Q}_1' = \frac{\vec{P}_1}{4} \qquad \vec{P}_1' = 4\vec{Q}_1'' = \vec{Q}_1 \vec{X} + \vec{Y}$$

$$\vec{X} = \frac{m_2(m_1+m_3)}{M}\frac{P_2^2}{4} - \frac{2 G m_1 m_2}{|\vec{R}_1 - \vec{R}_2|}Q_2^2 - 2E Q_2^2$$

$$\vec{Y} = -\frac{m_1 m_2}{M}\frac{L^T(\vec{P}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$\boxed{\vec{Q}_1'' - \omega^2 \vec{Q}_1 = \frac{1}{4}\vec{Y}} \qquad \omega^2 = \frac{1}{4}\vec{X}$$

Harmonic Oscillator with external quasi-periodic triggering $\Rightarrow$ resonances, strongly chaotic system (double pendulum!)

More Regularizations?

- 3-body

Aarseth 85: $$dt = \frac{R_1 R_2}{\sqrt{R_1 + R_2}} ds$$

- N-body

Heggie 74: $$dt = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} R_{ij} \, ds$$

$$R_{ij} = |\vec{r}_i - \vec{r}_j| = L(\vec{Q}_{ij})\vec{Q}_{ij}$$

$$\vec{V}_{ij} = \dot{\vec{R}}_{ij} = 2L(\vec{Q}_{ij})\vec{P}_{ij} / 4Q_{ij}^2$$

Mikkola 97

$$H(\vec{V}_{ij}, \vec{R}_{ij}) = \sum_{i<j} \frac{V_{ij}^2}{2\mu_{ij}} + \ldots + \sum_{i<j} \frac{G m_i m_j}{R_{ij}}$$

$$\Gamma = \left(H(V_{ij}, R_{ij}) - E \right) \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} R_{ij}$$

$$= \left[H\left(\frac{L(Q_{ij}) P_{ij}}{2 Q_{ij}^2}, L(Q_{ij}) Q_{ij} \right) - E \right] \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} Q_{ij}^2$$

Heggie's Global Regularization

Very Complicated Equations of Motion!

High Dimensionality: $N(N-1)/2$

- Chain Method (Mikkola)

$\vec{R}_K = \vec{r}_{K+1} - \vec{r}_K \qquad \vec{R}_K = L(\vec{Q}_K)\vec{Q}_K$

$$dt = \frac{1}{T+U}\, ds$$

- Slow-Down Treatment Mikkola, Aarseth 96

$$\ddot{\vec{r}} = - \frac{G m_1 m_2}{\kappa^2 |\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) + \vec{F}_{ext}$$

Slow-Down coefficient κ

$$\dot{\vec{r}} = \frac{1}{\kappa} \cdot \vec{v}$$

Period is κ-fold longer!

- Stumpff-Functions

$$\vec{u}'' + \frac{|h|}{2\mu} \vec{u} = \frac{u^2}{2} L^T(\vec{u}) \vec{F}_{ext} = \frac{u^2}{2} \cdot$$

Use Stumpff functions for series evaluation of solution. See later...

Examples of the regularizing transformations for special orbits

i) circular motion $\dot{r} = 0$

$$c^2 = r^2\dot{\varphi}^2 = r^2 v_\varphi^2 = \frac{G m_1 m_2}{\mu r^2} \cdot r^3 \implies$$

$$v_\varphi^2 = \frac{G m_1 m_2}{\mu r} = \frac{G(m_1 + m_2)}{r}$$

$$h = \frac{1}{2}\mu v_\varphi^2 - \mu v_\varphi^2 = -\frac{1}{2}\mu v_\varphi^2 = -\frac{G(m_1 + m_2)}{2r}$$

Reg. Transformation 1:

$$d\varphi = \frac{c}{r^2}\,dt = \frac{v_\varphi}{r}\,dt = \sqrt{G(m_1+m_2)} \cdot r^{-3/2}\,dt$$

φ: true anomaly, „time dilatation" for $r \to 0$

Reg. Transformation 2:

$$du = \sqrt{\frac{2|h|}{\mu}}\,\frac{dt}{r} = \frac{v_\varphi}{r}\,dt = \sqrt{G(m_1+m_2)} \cdot r^{-3/2}\,dt$$

For circular orbit same!

u: eccentric anomaly, fictitious time

ii) radial motion $c = 0$, $v_r = \mathring{r}$

$$\mathring{v}_r = -\frac{G m_1 m_2}{\mu r^2}$$

$$h = \frac{1}{2}\mu v_r^2 - \frac{G m_1 m_2}{r} = \text{const.} = h_0$$

<u>Reg. Transformation 1:</u>

not defined for $c = 0$

<u>Reg. Transformation 2:</u>

$$du = \sqrt{\frac{2 G m_1 m_2}{\mu r} - v_r^2}\,\frac{dt}{r} = \sqrt{\frac{2|h_0|}{\mu}}\,\frac{dt}{r}$$

still well defined, but time dilatation $\to \infty$ for $r \to 0$

well suited for eccentric motion!

Levi-Civita - Transformation

Why? Regularization interesting for perturbed 2-body motion (by 3rd body). $\Rightarrow$ Motion not confined in plane!

- Can we find regularization of full 2D-vector equation of motion? Levi-Civita
- Can we find regularization of full 3D-vector equation of motion?

Kustaanheimo + Stiefel, generalized Levi-Civita

2D - Vector Equations of Motion

$$\ddot{\vec{r}} = - \frac{G m_1 m_2}{\mu r^3} \vec{r}$$

$$\mu \frac{\dot{\vec{r}}^2}{2} - \frac{G m_1 m_2}{r} = h = \text{const.}$$

Start with regularization of eccentric anomaly type:

$$ds = \frac{dt}{r} \qquad \frac{1}{r}\frac{d}{ds} = \frac{d}{dt} \qquad \text{from } (*):$$

$$\frac{1}{r}\frac{d}{ds}\frac{1}{r}\frac{d}{ds}\vec{r} + \frac{Gm_1m_2}{\mu r^3}\vec{r} = 0$$

$$\frac{1}{r}\left(\frac{\vec{r}''}{r} - \frac{r'}{r^2}\vec{r}'\right) + \frac{Gm_1m_2}{\mu r^3}\vec{r} = 0$$

$$\boxed{\vec{r}'' - \frac{r'}{r}\vec{r}' + \frac{Gm_1m_2}{\mu r}\vec{r} = 0} \qquad (3)$$

from $(**)$:

$$\frac{1}{r^2}\left(\frac{d}{ds}\vec{r}\right)^2 - \frac{2Gm_1m_2}{\mu r} = 2h/\mu$$

$$\boxed{\frac{\vec{r}'^2}{r^2} - \frac{2Gm_1m_2}{\mu r} = 2h/\mu} \qquad (4)$$

We are not done since $r' \cdot \vec{r}' \neq \vec{r}'^2$

Search for Transformation 2D

- $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ real space

  mapping from 2D vector space into 2D v.sp.

- should be conformal

$$du^2 = \alpha \, dx^2 \qquad \alpha \neq 0$$

(different orbits should not be mapped onto each other!)

$$\Rightarrow \quad L(\vec{u})\,\vec{u} = \vec{x}$$

Matrix $L(u)$ orthogonal!

- further condition:

$$L(\vec{u})\,\vec{u} \mathrel{\hat{=}} \vec{u}^{\,2} = \vec{u} \cdot \vec{u}$$

with $\cdot$ = product in field („Körper")

Note 2D: $\cdot \Rightarrow$ commutative product in field $\mathbb{C}$ (complex numbers)

4D: $\cdot \Rightarrow$ non-commutative product $\mathbb{Q}$ Quaternions (Schiefkörper)

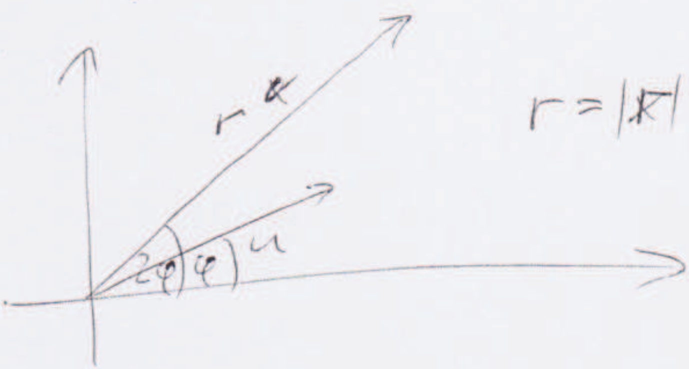
8D: $\cdot \Rightarrow$ „nullteilerfreies" product Cayley's numbers

Ansatz:

$L(\vec{u})\cdot\vec{u} = \vec{u}^2$ product of complex numbers

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 + i u_2$ $\qquad i^2 = -1$

$\vec{u}^2 = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{pmatrix}$ $\qquad \begin{matrix} \mathrm{Re}\, u^2 \\ \mathrm{Im}\, u^2 \end{matrix}$



$r = |x| = |u|^2$

$L(\vec{u}) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$

$$x_1 = u_1^2 - u_2^2$$
$$x_2 = 2u_1u_2$$

Levi-Civita transformation

$$x_1' = 2u_1u_1' - 2u_2u_2'$$
$$x_2' = 2u_1u_2' + 2u_1'u_2$$

$\vec{r} = L(\vec{u})\vec{u}$ $\qquad \vec{r}\,' = 2L(\vec{u})\vec{u}\,'$

$$ds^2 = dx_1^2 + dx_2^2 = \left(2u_1\,du_1 - 2u_2\,du_2\right)^2 + \left(2u_1\,du_2 + 2u_2\,du_1\right)^2$$

$$= 4\left(u_1^2 + u_2^2\right)\left(du_1^2 + du_2^2\right)$$

conformal mapping!

$u_1^2 + u_2^2 = 0$ if and only if

$x_1^2 + x_2^2 = 0$!

$\Rightarrow$ therefore no divisors of zero may exist in algebra!

$$\boxed{\Rightarrow r'^2 = 4u^2 u'^2}$$

$$\vec r = L(\vec u)\,\vec u = (\vec u \cdot \vec u)$$

$$\vec r\,' = 2L(\vec u)\,\vec u\,'$$

$$\vec r\,'' = 2L(\vec u\,')\,\vec u\,' + 2L(\vec u)\,\vec u\,''$$

since $L(\vec u)$ linear, homogeneous in $\vec u$

$$L^T(\vec u)\,L(\vec u) = |\vec u|^2 \cdot E = |\vec u|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L^T(\vec u) = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix} \Longrightarrow$$

$$L^{-1}(\vec u) = \frac{1}{|\vec u|^2}\, L^T(\vec u)$$

$r'^2 = |\vec r\,'|^2 = 4|\vec u|^2 |\vec u\,'|^2$; from (4):

$$\frac{4|\vec u|^2 (\vec u\,' \cdot \vec u\,')}{|\vec u|^4} - \frac{2Gm_1m_2}{\mu |\vec u|^2} = \frac{2h}{\mu}$$

$$(\vec u\,' \cdot \vec u\,') = \frac{Gm_1m_2}{2\mu} + \frac{h}{2\mu}|\vec u|^2$$

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from (3):

$$2L(\vec{u})\vec{u}'' + 2L(\vec{u}')\vec{u}' - \frac{r'}{|\vec{u}|^2}\cdot 2L(\vec{u})\vec{u}'$$

$$+ \frac{Gm_1m_2}{\mu|\vec{u}|^2} L(\vec{u})\vec{u} = 0 \quad \Big| \cdot \tfrac{1}{2}L^{-1}(\vec{u})$$

$$\boxed{\vec{u}'' + \frac{Gm_1m_2}{2\mu|\vec{u}|^2}\vec{u} - \frac{r'}{|\vec{u}|^2}\vec{u}' + \frac{L^T(\vec{u})L(\vec{u}')}{|\vec{u}|^2}\vec{u}' = 0}$$

$$\boxed{\vec{u}'' + \frac{Gm_1m_2}{2\mu|\vec{u}|^2}\vec{u} - \frac{\vec{u}'\cdot\vec{u}'}{|\vec{u}|^2}\vec{u} = 0}$$

to be demonstrated: $(\vec{u}'\cdot\vec{u}') = L(\vec{u}')\vec{u}'$

$$\boxed{\vec{u}'' - \frac{h}{2\mu}\vec{u} = 0}$$

2D periodic motion with period

$$\omega^2 = -\frac{h}{2\mu}$$