Post-Newtonian N-body Codes

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Introduction

N-body tools

Three-body formulation

PN implementations

Numerical examples

Discussion

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\, t + \frac{1}{2}\mathbf{F}_0^{(2)}\, t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\, t^3 \\
\mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\, t + \frac{1}{2}\mathbf{F}_0^{(3)}\, t^2
\end{aligned}$$

Prediction

$$\begin{aligned}
\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}_0^{(1)}\, \delta t'_j + \frac{1}{2}\mathbf{F}_0)\, \delta t'_j + \mathbf{v}_0)\, \delta t'_j + \mathbf{r}_0 \\
\mathbf{v}_j &= ((\frac{1}{2}\mathbf{F}_\mathbf{0}^{(1)}\, \delta t'_j + \mathbf{F}_0)\, \delta t'_j + \mathbf{v}_0\,; \qquad \delta t'_j = t - t_0
\end{aligned}$$

New forces $\qquad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\begin{aligned}
\mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{6}{t^3} \\
\mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{2}{t^2}
\end{aligned}$$

Corrector for i

$$\begin{aligned}
\Delta \mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)} \Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\, \Delta t^5 \\
\Delta \mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)} \Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\, \Delta t^4
\end{aligned}$$

AC Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n << \Delta t_d, \qquad n << N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Basic KS Relations

New coordinates $R = u_1^2 + u_2^2 + u_3^2 + u_4^2$

Time transformation $dt = R\,d\tau$

Coordinate transformation $\mathbf{R} = \mathcal{L}(\mathbf{u})\,\mathbf{u}$

Generalized Levi–Civita matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix}$$

Inverse relations

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ R_2 &= 2(u_1 u_2 - u_3 u_4) \\ R_3 &= 2(u_1 u_3 + u_2 u_4) \\ R_4 &= 0 \end{aligned}$$

Velocity $\dot{\mathbf{R}} = \dfrac{2\mathcal{L}\mathbf{u}'}{R}, \qquad R' = t'' = 2\mathbf{u}\cdot\mathbf{u}'$

Equations of motion

$$\begin{aligned} \mathbf{u}'' &= \frac{1}{2}\,h\,\mathbf{u} + \frac{1}{2}\,R\,\mathcal{L}^{\mathcal{T}}\,\mathbf{P} \\ h' &= 2\,\mathbf{u}'\cdot\mathcal{L}^{\mathcal{T}}\,\mathbf{P} \\ t' &= \mathbf{u}\cdot\mathbf{u} \end{aligned}$$

KS Decision-Making

Close encounter	$R_{\rm cl} = \dfrac{4\,r_{\rm h}}{N\,C^{1/3}}, \quad \Delta t_{\rm cl} = \beta \left(\dfrac{R_{\rm cl}^3}{\bar{m}}\right)^{1/2}$
Time-step criterion	$\Delta t_k < \Delta t_{\rm cl}$
Neighbour list search	list all $r_{kj}^2, \quad \Delta t_j < 2\,\Delta t_{\rm cl}$
Two-body selection	$R_{kl} < R_{\rm cl}, \quad \dot{R}_{kl} < 0$
Dominant motion	$\dfrac{m_k + m_l}{R_{kl}^2} > \dfrac{m_k + m_j}{R_{kj}^2}$
KS initialization	$\mathbf{F}_U,\ \mathbf{F}'_U,\ \Delta\tau\ \&\ t^{(n)} \ \Rightarrow\ \Delta t$
Initialization of c.m.	$\mathbf{r}_{\rm cm} = \dfrac{m_k\,\mathbf{r}_k + m_l\,\mathbf{r}_l}{m_k + m_l}$
Perturber search	$r_{\rm p} < \left(\dfrac{2m_{\rm p}}{m_{\rm b}\gamma_{\rm min}}\right)^{1/3} a\,(1+e)$
Slow-down adjustment	$\gamma < \gamma_0, \quad \Delta\tau \Rightarrow \kappa\,\Delta t$
Termination test	$R > R_0, \quad \gamma > \gamma^*$
Delayed termination	$T_{\rm block} - t > \Delta t_i$
Final iteration	$\Delta\tau$ from $\dot{\tau},\ \ddot{\tau}, \ldots$ and δt
Polynomial initialization	$\mathbf{F}_j,\ \dot{\mathbf{F}}_j,\ \Delta t_j, \quad j = k,\ l$

Hermite KS with block-steps

Predict $\mathbf{u},\ \mathbf{u}',\ \mathbf{h}$ to highest order

$$
\begin{aligned}
\boldsymbol{u}_{\text{pred}} &= \boldsymbol{u} + \Delta\tau\boldsymbol{u}' + \frac{\Delta\tau^2}{2}\boldsymbol{u}'' + \frac{\Delta\tau^3}{6}\boldsymbol{u}''' + \frac{\Delta\tau^4}{24}\boldsymbol{u}^{\langle 4\rangle} + \frac{\Delta\tau}{12} \\
\boldsymbol{u}'_{\text{pred}} &= \boldsymbol{u}' + \Delta\tau\boldsymbol{u}'' + \frac{\Delta\tau^2}{2}\boldsymbol{u}''' + \frac{\Delta\tau^3}{6}\boldsymbol{u}^{\langle 4\rangle} + \frac{\Delta\tau^4}{24}\boldsymbol{u}^{\langle 5\rangle}, \\
h_{\text{pred}} &= h + \Delta\tau h' + \frac{\Delta\tau}{2}h'' + \frac{\Delta\tau}{6}h''' + \frac{\Delta\tau}{24}h''''.
\end{aligned}
$$

Force evaluation

$$
\begin{aligned}
\boldsymbol{P} &= \mathcal{L}_{\boldsymbol{u}}^T\boldsymbol{F}, \\
\boldsymbol{P}' &= \mathcal{L}_{\boldsymbol{u}'}^T\boldsymbol{F} + r\mathcal{L}_{\boldsymbol{u}}^T\dot{\boldsymbol{F}}.
\end{aligned}
\tag{2}
$$

Derivatives for corrector

$$
\begin{aligned}
\boldsymbol{u}'' &= \frac{1}{2}(h\boldsymbol{u} + r\boldsymbol{P}), \\
\boldsymbol{u}''' &= \frac{1}{2}(h'\boldsymbol{u} + h\boldsymbol{u}' + r'\boldsymbol{P} + r\boldsymbol{P}'), \\
h' &= 2(\boldsymbol{u}'\cdot\boldsymbol{P}), \\
h'' &= 2(\boldsymbol{u}''\cdot\boldsymbol{P} + \boldsymbol{u}'\cdot\boldsymbol{P}').
\end{aligned}
\tag{3}
$$

Fourth-order corrector (Hut, Makino & McMillan 1995)

$$\begin{aligned}
\boldsymbol{u}_{\mathrm{corr}} &= \boldsymbol{u}_{\mathrm{old}} + \frac{\Delta\tau}{2}(\boldsymbol{u}'_{\mathrm{corr}} + \boldsymbol{u}'_{\mathrm{old}}) - \frac{\Delta\tau^2}{12}(\boldsymbol{u}''_{\mathrm{new}} - \boldsymbol{u}''_{\mathrm{old}}), \\
\boldsymbol{u}'_{\mathrm{corr}} &= \boldsymbol{u}'_{\mathrm{old}} + \frac{\Delta\tau}{2}(\boldsymbol{u}''_{\mathrm{new}} + \boldsymbol{u}''_{\mathrm{old}}) - \frac{\Delta\tau^2}{12}(\boldsymbol{u}'''_{\mathrm{new}} - \boldsymbol{u}'''_{\mathrm{old}}), \\
h_{\mathrm{corr}} &= h_{\mathrm{old}} + \frac{\Delta\tau}{2}(h'_{\mathrm{new}} + h'_{\mathrm{old}}) - \frac{\Delta\tau^2}{12}(h''_{\mathrm{new}} - h''_{\mathrm{old}}).
\end{aligned} \tag{4}$$

Velocity derivatives for prediction

$$\begin{aligned}
\boldsymbol{u}^{\langle 4\rangle}_{\mathrm{mid}} &= \frac{\boldsymbol{u}'''_{\mathrm{new}} - \boldsymbol{u}'''_{\mathrm{old}}}{\Delta\tau}, \\
\boldsymbol{u}^{\langle 5\rangle}_{\mathrm{mid}} &= \frac{12}{\Delta\tau^2}\left(\frac{\boldsymbol{u}'''_{\mathrm{new}} + \boldsymbol{u}'''_{\mathrm{old}}}{2} - \frac{\boldsymbol{u}''_{\mathrm{new}} - \boldsymbol{u}''_{\mathrm{old}}}{\Delta\tau}\right).
\end{aligned} \tag{5}$$

Shifted from midpoint

$$\begin{aligned}
\boldsymbol{u}^{\langle 4\rangle}_{\mathrm{new}} &= \boldsymbol{u}^{\langle 4\rangle}_{\mathrm{mid}} + \frac{\Delta\tau}{2}\boldsymbol{u}^{\langle 5\rangle}_{\mathrm{mid}}, \\
\boldsymbol{u}^{\langle 5\rangle}_{\mathrm{new}} &= \boldsymbol{u}^{\langle 5\rangle}_{\mathrm{mid}}, \\
h'''_{\mathrm{new}} &= h'''_{\mathrm{mid}} + \frac{\Delta\tau}{2}h''''_{\mathrm{mid}}, \\
h''''_{\mathrm{new}} &= h''''_{\mathrm{mid}}.
\end{aligned} \tag{6}$$

Taylor series for physical time

$$
\begin{aligned}
t' &= \boldsymbol{u} \cdot \boldsymbol{u}, \\
t'' &= 2(\boldsymbol{u} \cdot \boldsymbol{u}'), \\
t''' &= 2(\boldsymbol{u} \cdot \boldsymbol{u}'' + \boldsymbol{u}' \cdot \boldsymbol{u}'), \\
t^{\langle 4\rangle} &= 2(\boldsymbol{u} \cdot \boldsymbol{u}''') + 6(\boldsymbol{u}' \cdot \boldsymbol{u}''), \\
t^{\langle 5\rangle} &= 2(\boldsymbol{u} \cdot \boldsymbol{u}^{\langle 4\rangle} + 8(\boldsymbol{u}' \cdot \boldsymbol{u}''') + 6(\boldsymbol{u}'' \cdot \boldsymbol{u}''), \\
t^{\langle 6\rangle} &= 2(\boldsymbol{u} \cdot \boldsymbol{u}^{\langle 5\rangle} + 10(\boldsymbol{u}' \cdot \boldsymbol{u}^{\langle 4\rangle} + 20(\boldsymbol{u}'' \cdot \boldsymbol{u}'''). \qquad (7)
\end{aligned}
$$

Next look-up time

$$t_{\mathrm{next}} = t + \sum_{n=1}^{6} \frac{t^{\langle n\rangle} \Delta\tau^{n}}{n!}. \qquad (8)$$

Regularized time-step by inversion

$$-\Delta t + \sum_{n=1}^{6} \frac{t^{\langle n\rangle} \Delta\tau^{n}}{n!} = 0. \qquad (9)$$

Newton-Raphson iteration to solve for $\Delta\tau$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Initial guess

$$\Delta\tau_1 = \frac{\Delta t_{\mathrm{bs}}}{\Delta t}\Delta\tau_{\mathrm{orig}},$$

with quantized time-step $\Delta t_{\mathrm{bs}}(= 2^{-n} \leq \Delta t_{\mathrm{orig}})$.

Commensurability condition

$$\mathrm{mod}\ (t_{\mathrm{sys}}, \Delta t_{\mathrm{bs}}) = 0.$$

Solve equation (9) to obtain $\Delta\tau_{\mathrm{bs}}$.

Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 \ = \ -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} \ = \ \dfrac{m_k m_3}{m_k + m_3}$

$$H \ = \ \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS coordinate transformation $\mathbf{Q}_k^2 \ = \ R_k\,, \quad (k = 1, 2)$

Time transformation $dt \ = \ R_1 R_2 \, d\tau$

Regularized Hamiltonian $\Gamma^* \ = \ R_1 R_2 \, (H - E_0)$

$$\Gamma^* \ = \ \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_{3-k} \, \mathbf{P}_k^2 \ + \ \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2^T \mathbf{P}_2$$

$$- \, m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} \ = \ \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}\,, \qquad \frac{d\mathbf{P}_k}{d\tau} \ = \ -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \rightarrow 0$ or $R_2 \rightarrow 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| \ > \ \max\,(R_1,\ R_2)$

Three-Body Transformations

Coordinates & momenta $\mathbf{q}_k = \tilde{\mathbf{q}}_k - \tilde{\mathbf{q}}_3, \quad \mathbf{p}_k = \tilde{\mathbf{p}}_k$

Regularized coordinates ($q_1 \geq 0$)

$$\begin{aligned} Q_1 &= [\tfrac{1}{2}(|\mathbf{q}_1| + q_1)]^{1/2} \\ Q_2 &= \tfrac{1}{2} q_2 / Q_1 \\ Q_3 &= \tfrac{1}{2} q_3 / Q_1 \\ Q_4 &= 0 \end{aligned}$$

Regularized momenta $\mathbf{P}_k = \mathbf{A}_k \, \mathbf{p}_k$

Basic matrix

$$\mathbf{A}_1 = 2 \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ -Q_3 & -Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{bmatrix}$$

KS transformations $\mathbf{q}_k = \tfrac{1}{2} \mathbf{A}_k^{\mathrm{T}} \mathbf{Q}_k$

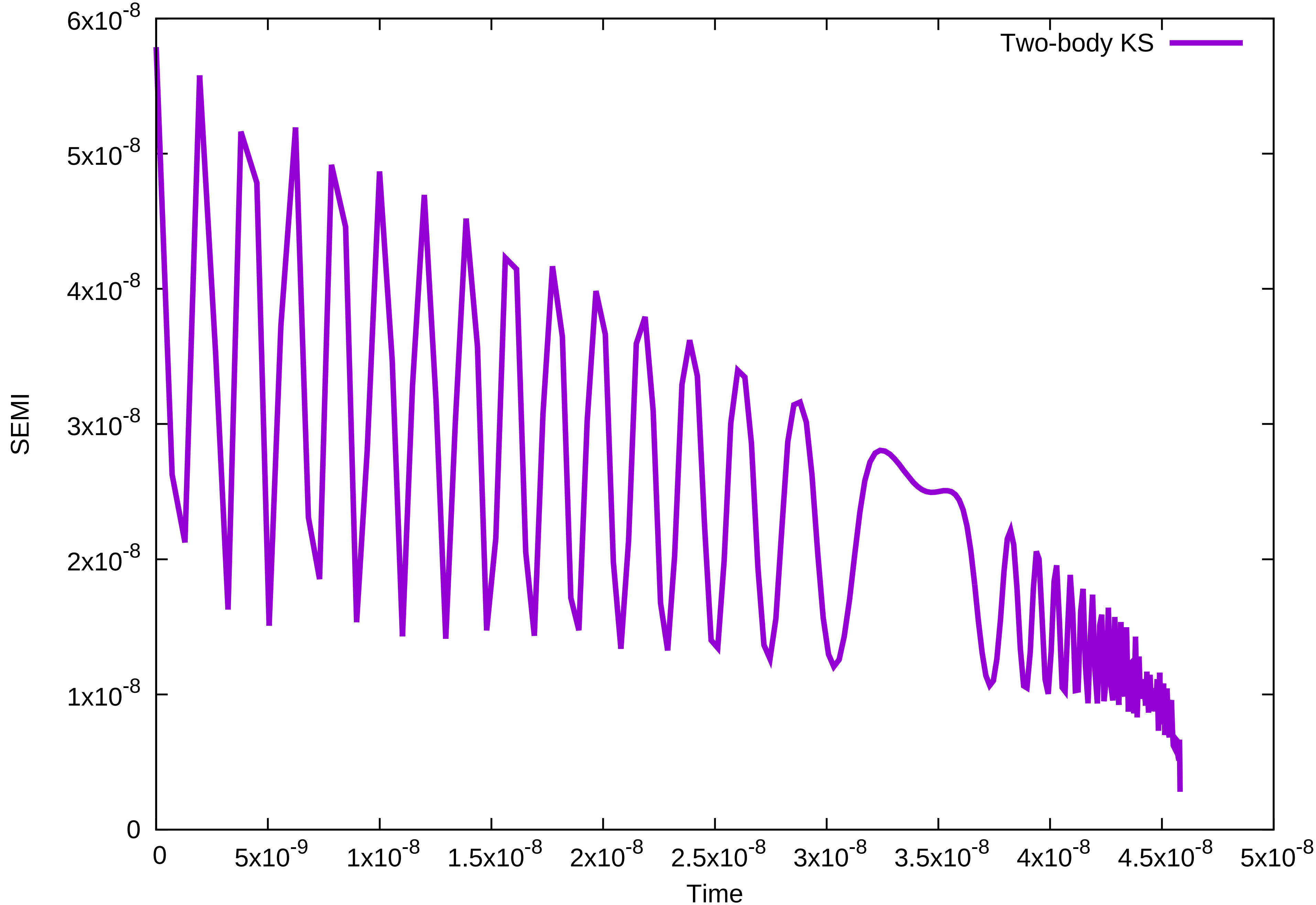
Physical momenta $\mathbf{p}_k = \tfrac{1}{4} \mathbf{A}_k^{\mathrm{T}} \mathbf{P}_k / R_k$

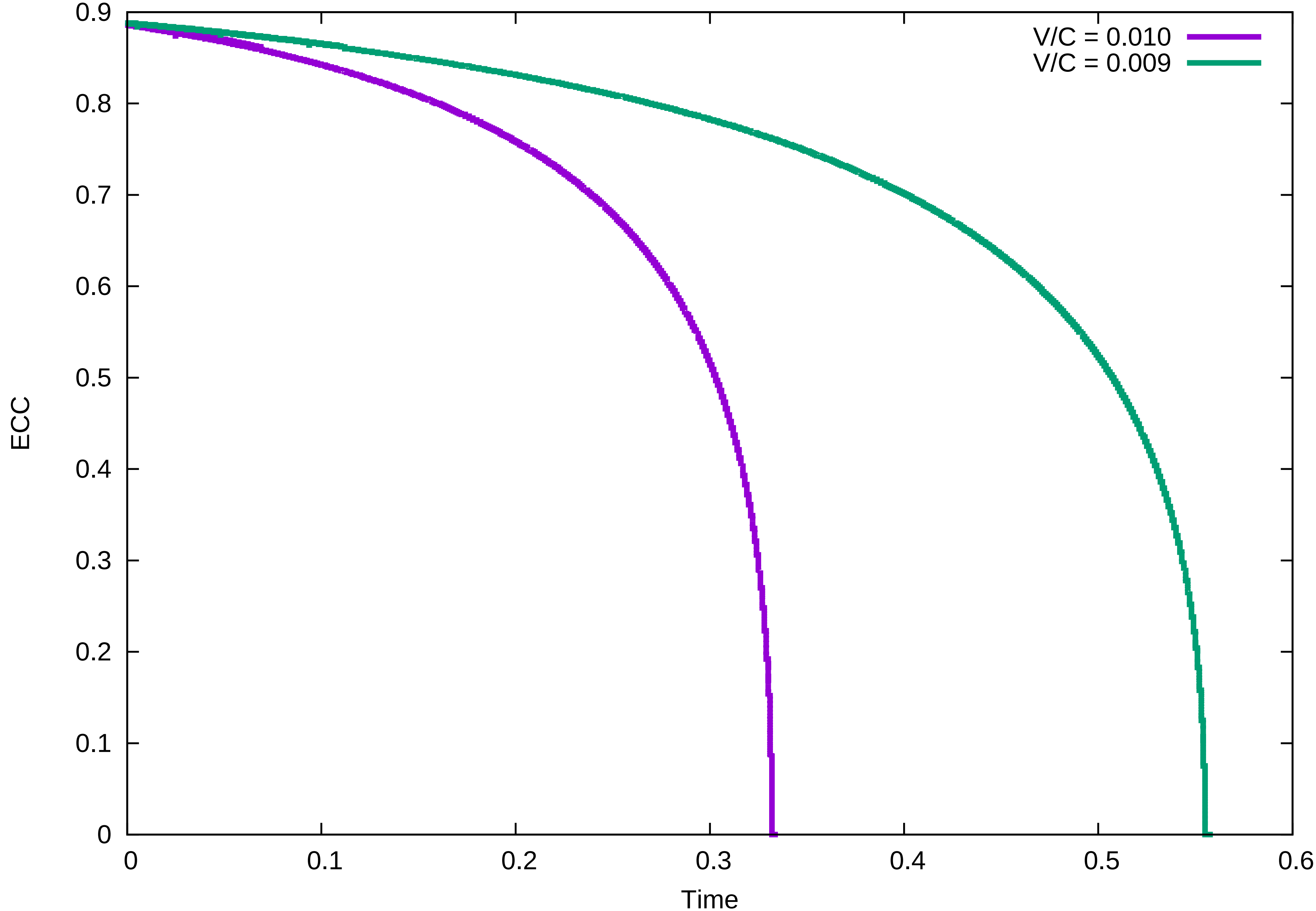
Coordinates & momenta

$$\begin{aligned} \tilde{\mathbf{q}}_3 &= -\sum_{k=1}^{2} m_k \mathbf{q}_k / M \\ \tilde{\mathbf{q}}_k &= \tilde{\mathbf{q}}_3 + \mathbf{q}_k \\ \tilde{\mathbf{p}}_k &= \mathbf{p}_k \\ \tilde{\mathbf{p}}_3 &= -(\mathbf{p}_1 + \mathbf{p}_2) \qquad (k = 1, 2) \end{aligned}$$

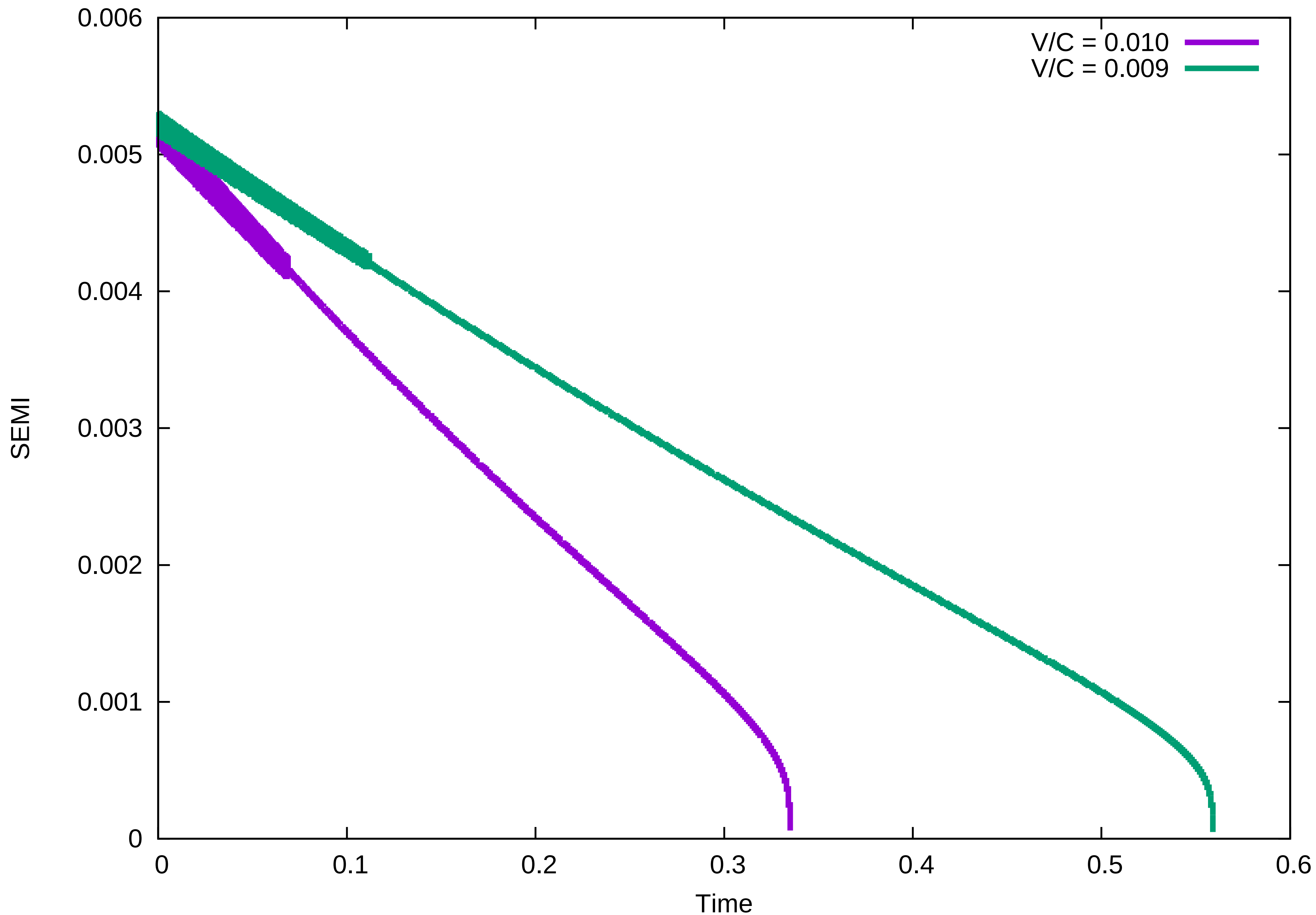
PN Codes

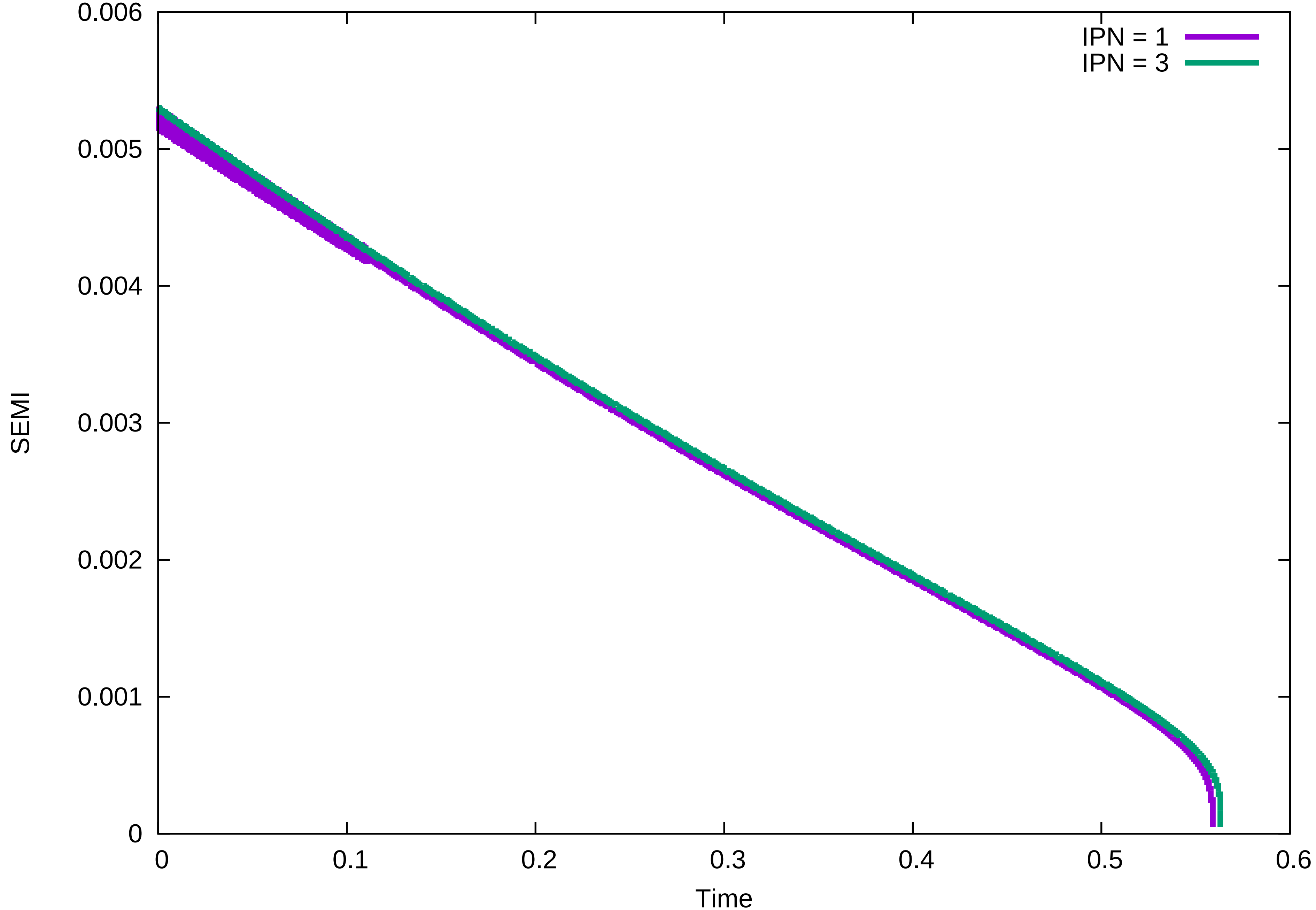
Two-body	Hermite KS toy code
	alternative PN formulations
Three-body	AZ regularization
	dominant binary, sequential terms
	homework: Einstein shift
Chain	Algorithmic regularization
	compact sub-system
	sequential PN, unperturbed KS


Two-body KS
SEMI
Time
6x10^-8
5x10^-8
4x10^-8
3x10^-8
2x10^-8
1x10^-8
0
5x10^-9
1x10^-8
1.5x10^-8
2x10^-8
2.5x10^-8
3x10^-8
3.5x10^-8
4x10^-8
4.5x10^-8
5x10^-8

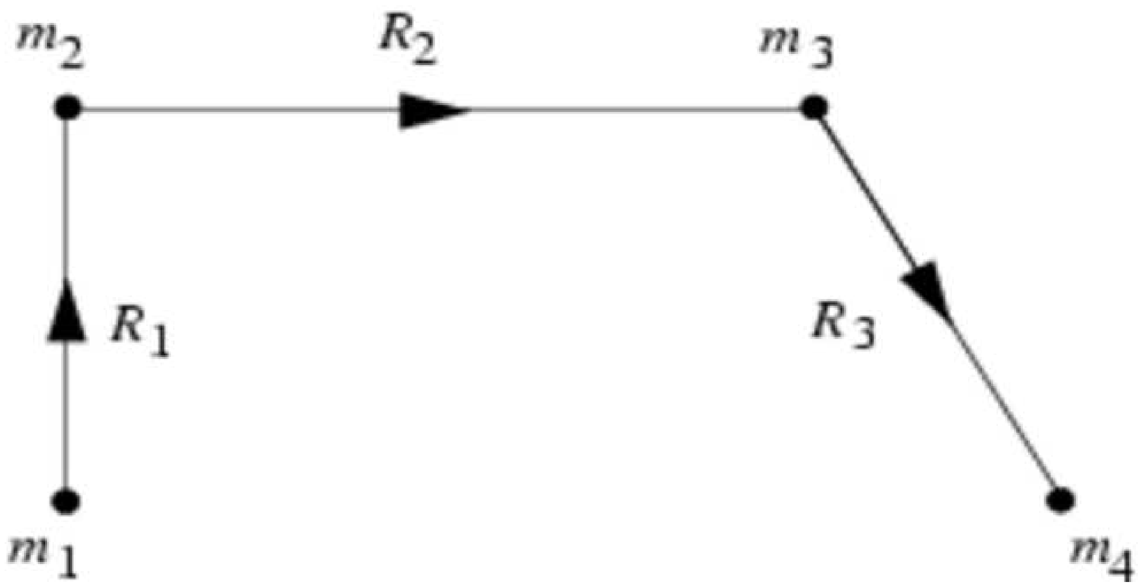


V/C = 0.010
V/C = 0.009
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ECC
0
0.1
0.2
0.3
0.4
0.5
0.6
Time


V/C = 0.010
V/C = 0.009
SEMI
Time
0.006
0.005
0.004
0.003
0.002
0.001
0
0
0.1
0.2
0.3
0.4
0.5
0.6



IPN = 1
IPN = 3
SEMI
Time
0.006
0.005
0.004
0.003
0.002
0.001
0
0
0.1
0.2
0.3
0.4
0.5
0.6



m_2
R_2
m_3
R_1
R_3
m_1
m_4

Initialization

Two-body	$R < R_{\mathrm{cl}}, \quad \dot{R} < 0, \quad \Rightarrow$ KS
Hierarchy	$B + S \quad \Rightarrow \quad H,$ Stability test
Triple chain	$a_1(1 - e_1) < 3a_0, \quad \dot{r} < 0, \quad \Rightarrow$ T
Four-body chain	$B + B, \quad \dot{r} < 0, \quad \Rightarrow$ B-B
Higher orders	$T + B, \quad \dot{r} < 0, \quad \Rightarrow \quad Q$

Chain Initialization

Unperturbed exit	$\Delta\omega \; > \; 1 \times 10^{-3}$
Activate chain	$N_{\rm ch} = 2$ or > 2
Alternative path	perturbed KS $\Rightarrow$ search
Selection	chain members & c.m.
Initialization	c.m. force & chain vectors
Synchronization	block time-step
Interface	perturber list

Termination

Hyperbolic KS	$R > R_0$ or $\gamma > 0.001$
Binary KS	$\gamma > 0.1, \quad \gamma = \dfrac{(\mathbf{F_1} - \mathbf{F_2})R^2}{m_1 + m_2}$
Collision	$KS \;\Rightarrow S$
Triple	$d_3 > 20a, \quad \Rightarrow B + S$
Hierarchy	$a_1(1 - e_1) < Sa_0, \quad \gamma > 0.01$
ARchain	$R_1 << R_2 + R_3 \quad \Rightarrow B + 2S$

Relativistic Effects

Einstein shift

$$\Delta\omega = \frac{6\pi M}{ac^2(1-e^2)}$$

GR radiation time-scale

$$t_{\rm GR} \; = \; \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_1^3}$$

$$X = \frac{m_2}{m_1}, \qquad g(e) \simeq \frac{(1-e^2)^{7/2}}{4.5}$$

Speed of light & Schwarzschild radius

$$c = \frac{3\times 10^5}{V^*}, \quad R_{\rm Sch} = \frac{2M}{c^2}$$

Velocities

$$V^* = \alpha\left(\frac{M}{R_{pc}}\right)^{1/2}, \quad v_{\rm max} = \left(\frac{M}{a}\right)^{1/2}\left(\frac{1+e}{1-e}\right)^{1/2}$$

PN conditions

$$\Delta\omega \; > \; 1\times 10^{-4}, \quad t_{\rm GR} \; < \; t^*, \qquad r \; < \; n\, R_{\rm Sch}$$

Post-Newtonian Terms

Equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession

$$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession

$$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation

$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation

$$\mathbf{P}_{GR} = \frac{M}{c^2r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

GR spin — dominant BH

ARC energy change

$$\Delta E = \frac{m_1 m_2}{M}\int(\mathbf{P}_{\rm GR} + \mathbf{P}_{\rm dyn}) \cdot \mathbf{v}\, dt$$

Decision-Making

Increasing GR effect	$t_{\rm GR} \; < \; 500, \;\; 50, \;\; 1$
	$\rm IPN \; = \; 1, \;\; 2, \;\; 3$
GR time-scale	$T_z = \dfrac{150 a^4}{M^3}(1-e^2)^{7/2}$ Myr
Graduated PN	$\Delta w > (1, 10, 100) \times 10^{-4} \quad \Rightarrow {\rm IPN} = 1, \;\; 2, \;\; 3$
Coalescence	$R \; < \; 4 R_{\rm Sch} \; = \; \dfrac{8\,M}{c^2}$
Alternative merging	${\rm IPN} = 3, \quad N = 2, \quad N_{\rm p} = 0$
	${\rm IPN} \geq 2, \quad a\,(1-e) < R_{\rm Sch}$
	${\rm IPN} = 3, \quad N = 3, \quad a_1\,(1-e_1) > 100\,a$
Unperturbed KS	$t_{\rm GR} < 500 \;\&\; \Delta w > 1 \times 10^{-4}, \quad \gamma < 1 \times 10^{-6}$
Tidal disruption	KS or CHAIN

PN Elements

Energy $$\epsilon_b = \epsilon_0 + \frac{\epsilon_1}{c^2} + \frac{\epsilon_2}{c^4} + \frac{\epsilon_3}{c^6}, \qquad a = -\frac{M}{2\epsilon_b}$$

$$\epsilon_0 = \frac{1}{2}V^2 - \frac{M}{R}, \qquad \eta = \frac{m_1 m_2}{M^2}$$

$$\epsilon_1 = \frac{1}{2}\frac{M}{R} + \frac{3}{8}(1-3\eta)V^4 + \frac{1}{2}\left((3+\eta)V^2 + \eta\dot{R}^2\right)\frac{M}{R}$$

Angular momentum $$\mathbf{J} = \mathbf{J}_0(1 + f_1/c^2 + f_2/c^4)$$

Eccentricity $$e^2 = (1 - \frac{\mathbf{J}^2}{Ma})$$

Lenz vector $$\mathbf{e} = \mathbf{V} \times \mathbf{R} \times \mathbf{V}/M - \mathbf{R}/R$$

Periapse advance $$\Delta\omega = \frac{6\pi M}{c^2 a(1-e^2)}$$

Kozai-Lidov cycles

$$T_{\mathrm{Kozai}} = \frac{T_{\mathrm{out}}^2}{T_{\mathrm{in}}}\left(\frac{1+q_{\mathrm{out}}}{q_{\mathrm{out}}}\right)(1-e_{\mathrm{out}}^2)^{3/2} g(e_{\mathrm{in}}, \omega_{\mathrm{in}}, \psi)$$

Unperturbed GR Orbit

Compact objects	$\max\,(K_1^*, K_2^*)\ >\ 12$
Derivatives	$\dot{a} = -\dfrac{64 M_1 M_2 (M_1 + M_2)}{5 c^5 a^3 (1 - e^2)^{7/2}}$
	$\dot{e} = -\dfrac{304 e M_1 M_2 (M_1 + M_2)}{15 c^5 a^4 (1 - e^2)^{5/2}} g(e)$
Einstein shift	$\Delta\omega = \dfrac{6\pi (M_1 + M_2)}{a c^2 (1 - e^2)}$
KS rotation	$\theta = \dfrac{\Delta\omega \Delta t}{T_{\rm K}}, \quad \text{rotate by } \theta/2$
New elements	$a_1 = a + \dot{a}\Delta t, \quad e_1 = \max\,(e + \dot{e}\Delta t, 0)$
Energy updating	$\Delta E = \mu (H_{\rm old} - H_{\rm new})$
Modify KS orbit	Shrink, $e = \text{const}$; decrease, $H = \text{const}$
Validity	$1 \times 10^{-4} < \Delta\omega < 1 \times 10^{-3}$
Unperturbed KS	$\gamma < 1 \times 10^{-6}$

Membership

Reduction escape R_1 or $R_{n-1} > R^*$, $\dot{r} > 0$

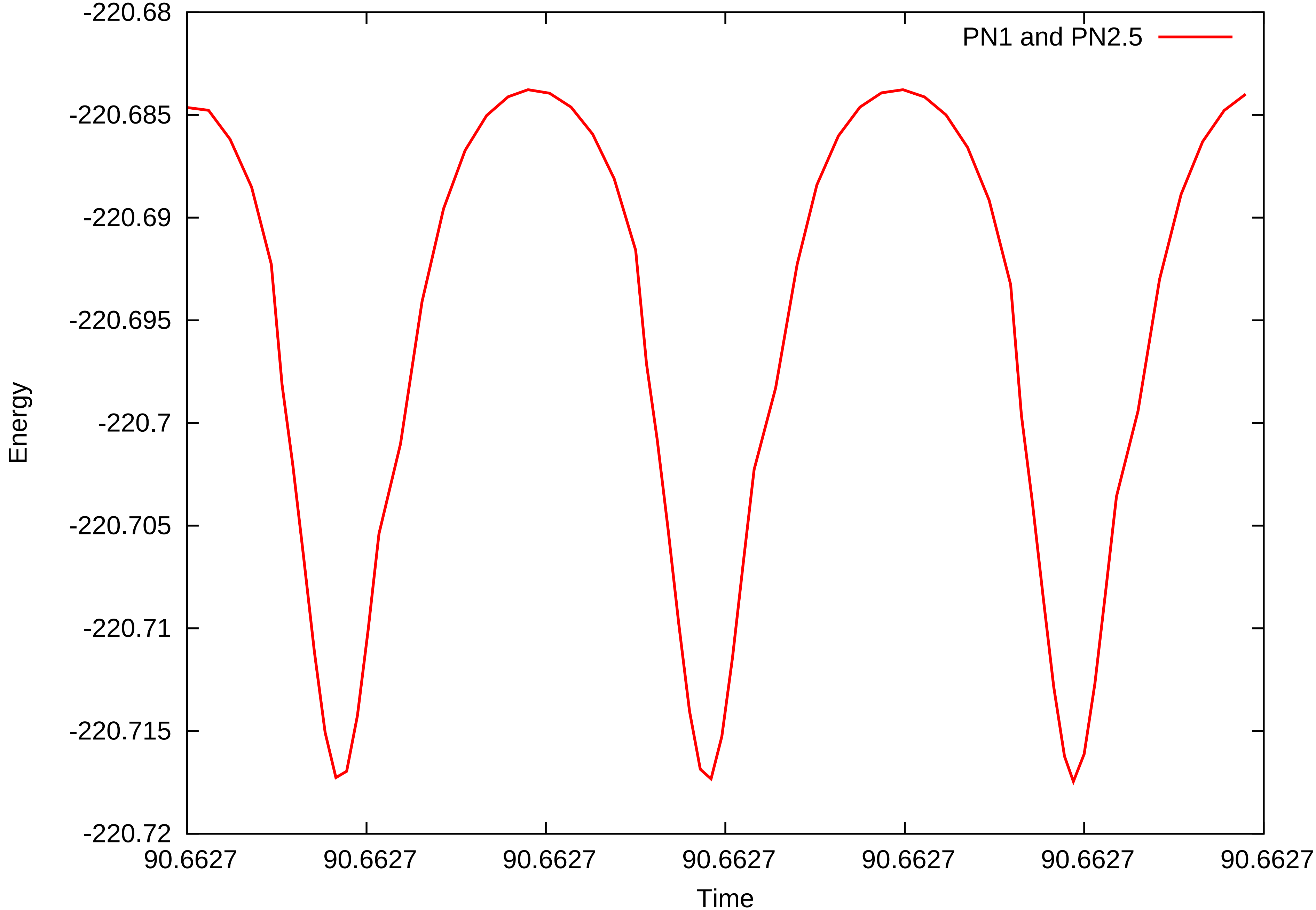
Double KS n = 4, two well separated pairs

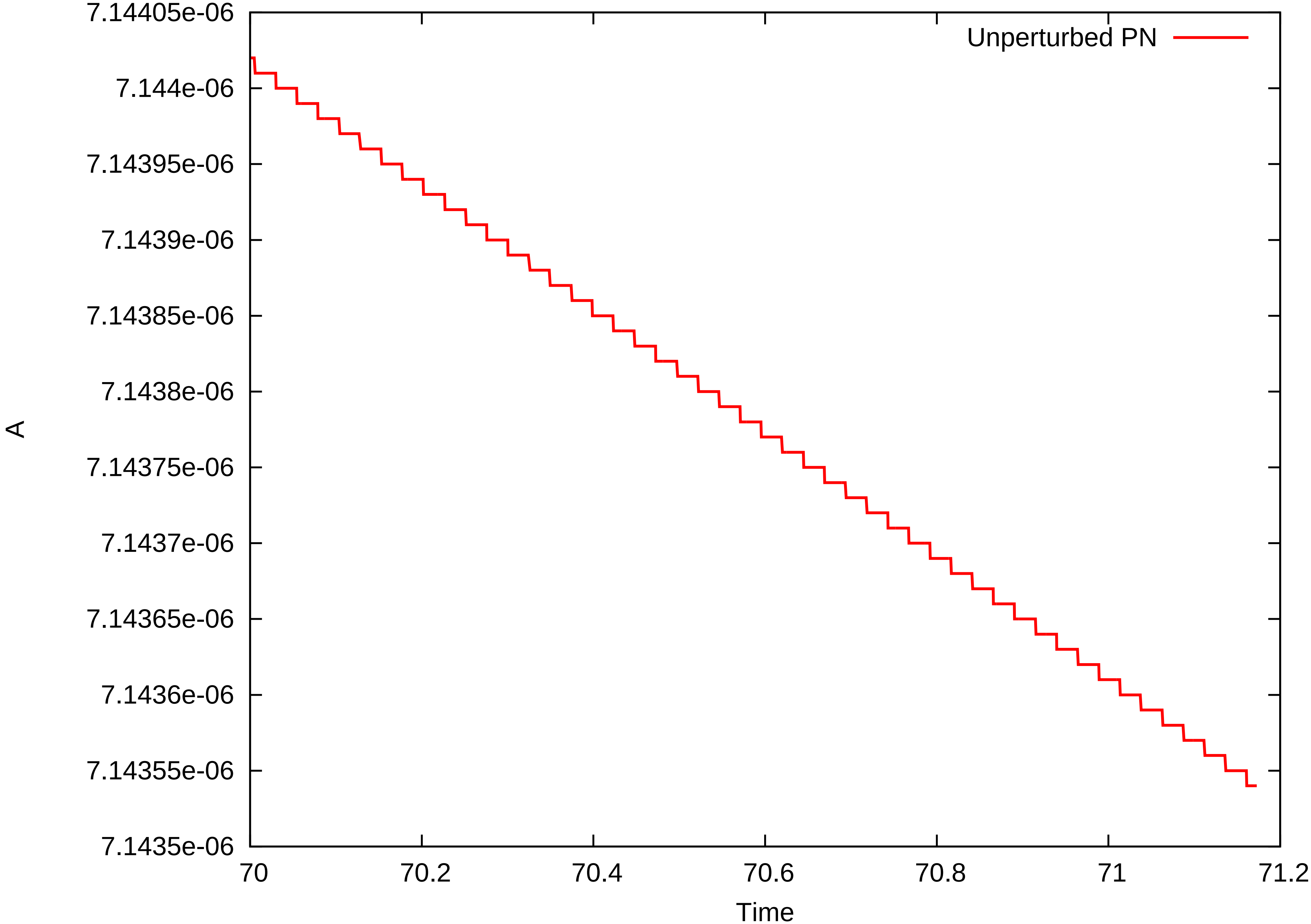
Injection $\gamma > 1 \times 10^{-2}$ & $r_{\rm p} < R^*$, $\dot{r}_{\rm p} < 0$

Retention $n = 2$, $IPN > 0$, $\Delta\omega > 1 \times 10^{-3}$

Perturbers massive binary, $N_{\rm p} \simeq 3 - 10$

Perturbation $\gamma \simeq \frac{2m_{\rm p}}{M_{\rm bh}} \max \left(\frac{R_k}{d_{\rm p}}\right)^3$

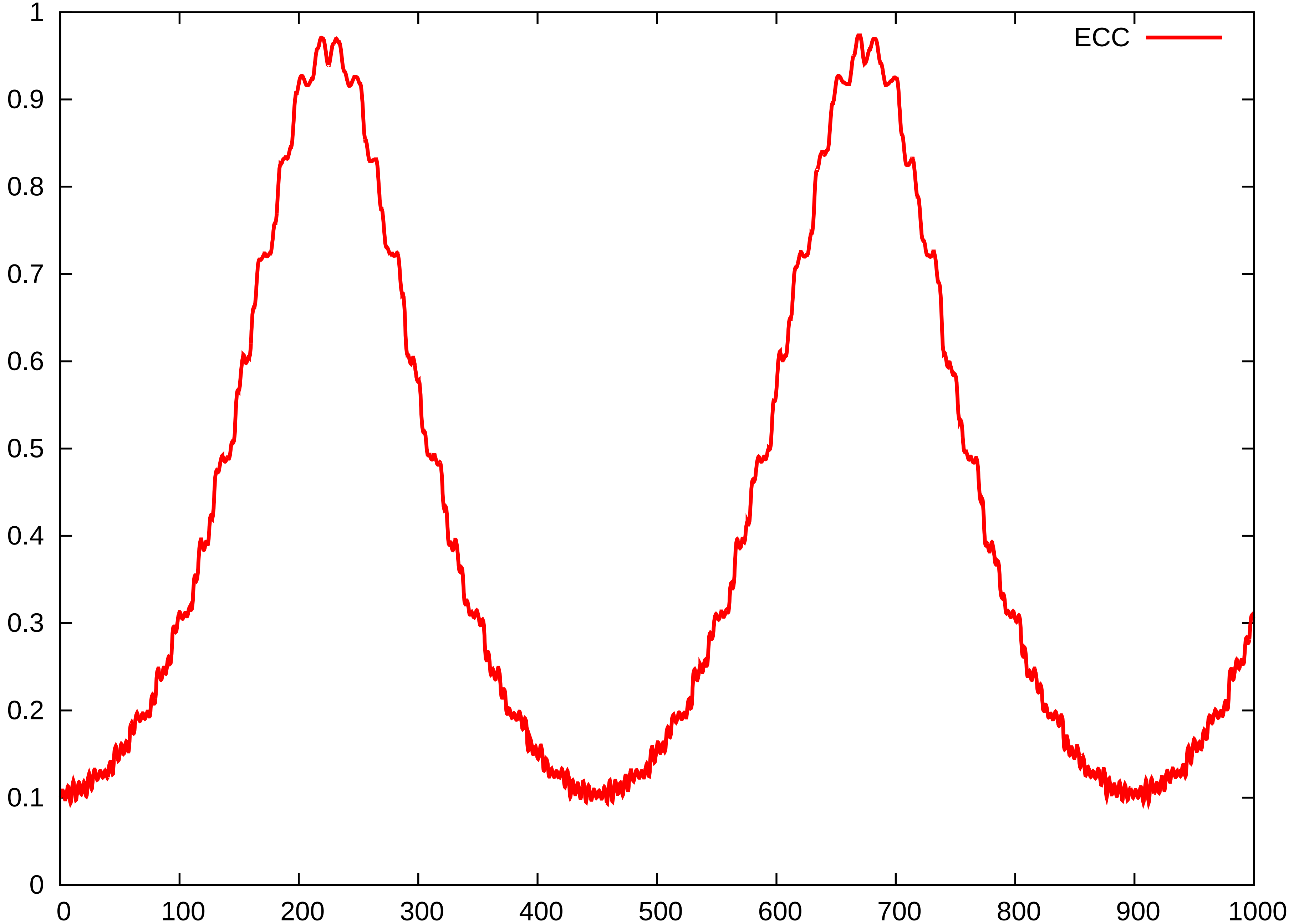

PN1 and PN2.5
Energy
Time
-220.68
-220.685
-220.69
-220.695
-220.7
-220.705
-220.71
-220.715
-220.72
90.6627
90.6627
90.6627
90.6627
90.6627
90.6627
90.6627



Unperturbed PN
A
Time
7.14405e-06
7.144e-06
7.14395e-06
7.1439e-06
7.14385e-06
7.1438e-06
7.14375e-06
7.1437e-06
7.14365e-06
7.1436e-06
7.14355e-06
7.1435e-06
70
70.2
70.4
70.6
70.8
71
71.2

Energy Corrections

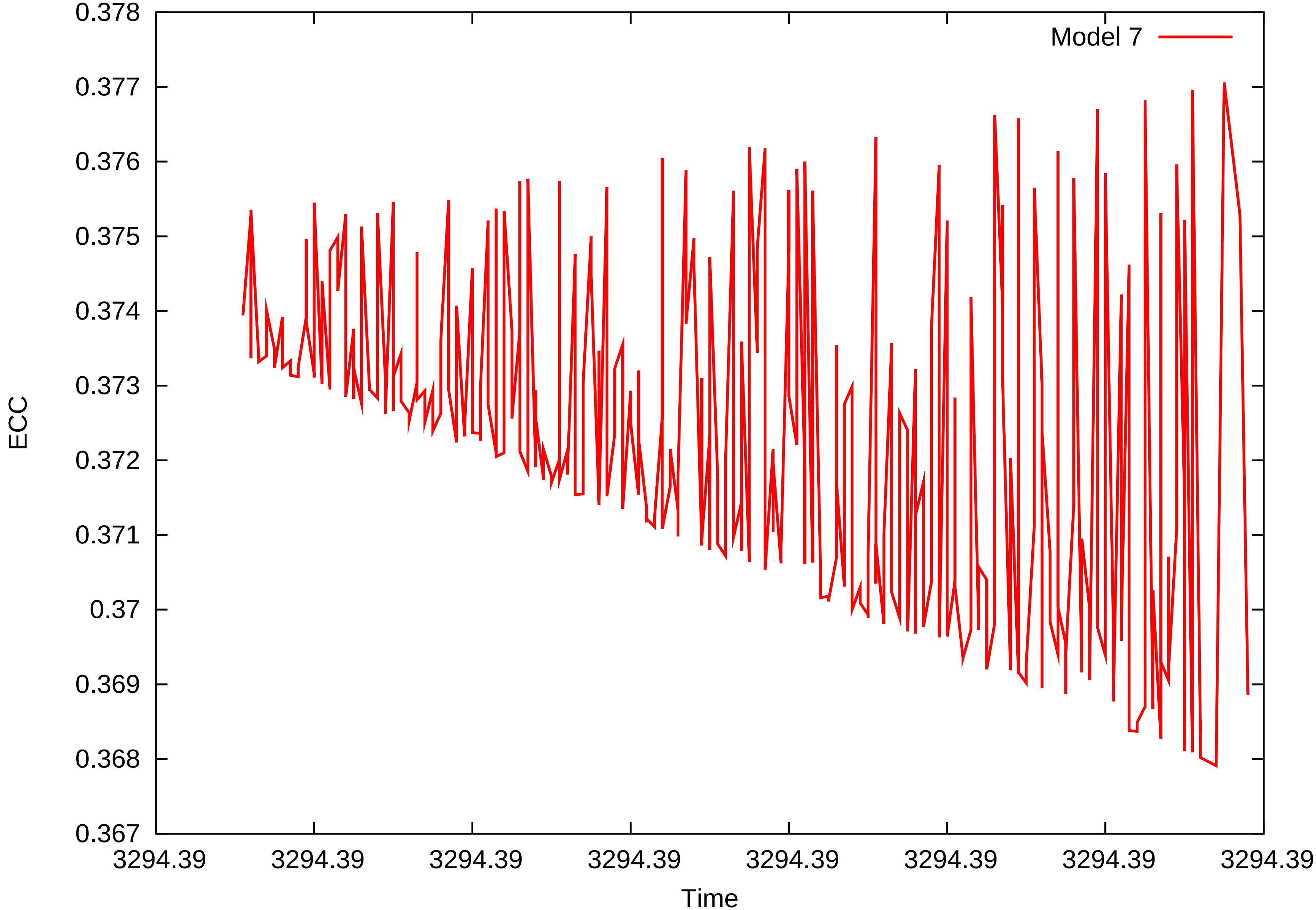
Unperturbed KS	$1 \times 10^{-4} \ < \ \Delta\omega \ < \ 1 \times 10^{-3}$
Perturbed KS	$\mathbf{P_{2.5}}$ & $\dot{\mathbf{P}}_{\mathbf{2.5}}$ added to $\mathbf{F}$ & $\dot{\mathbf{F}}$
Energy derivatives	$\dot{E} = \mathbf{P} \cdot \mathbf{v}, \quad \ddot{\mathbf{E}} = \mathbf{P} \cdot \dot{\mathbf{v}} + \dot{\mathbf{P}} \cdot \mathbf{v}$
Hermite method	$\Delta h = \frac{1}{2}(\dot{E}_0 + \dot{E})\Delta t + \frac{1}{12}(\ddot{E}_0 - \ddot{E})\Delta t^2$
Energy loss	$\sum \mu \Delta h$
Conservation	$E_{\rm ch} = E_{\rm sub} - \sum \Delta E$ $E_{\rm tot} = E_N + E_{\rm ch}$

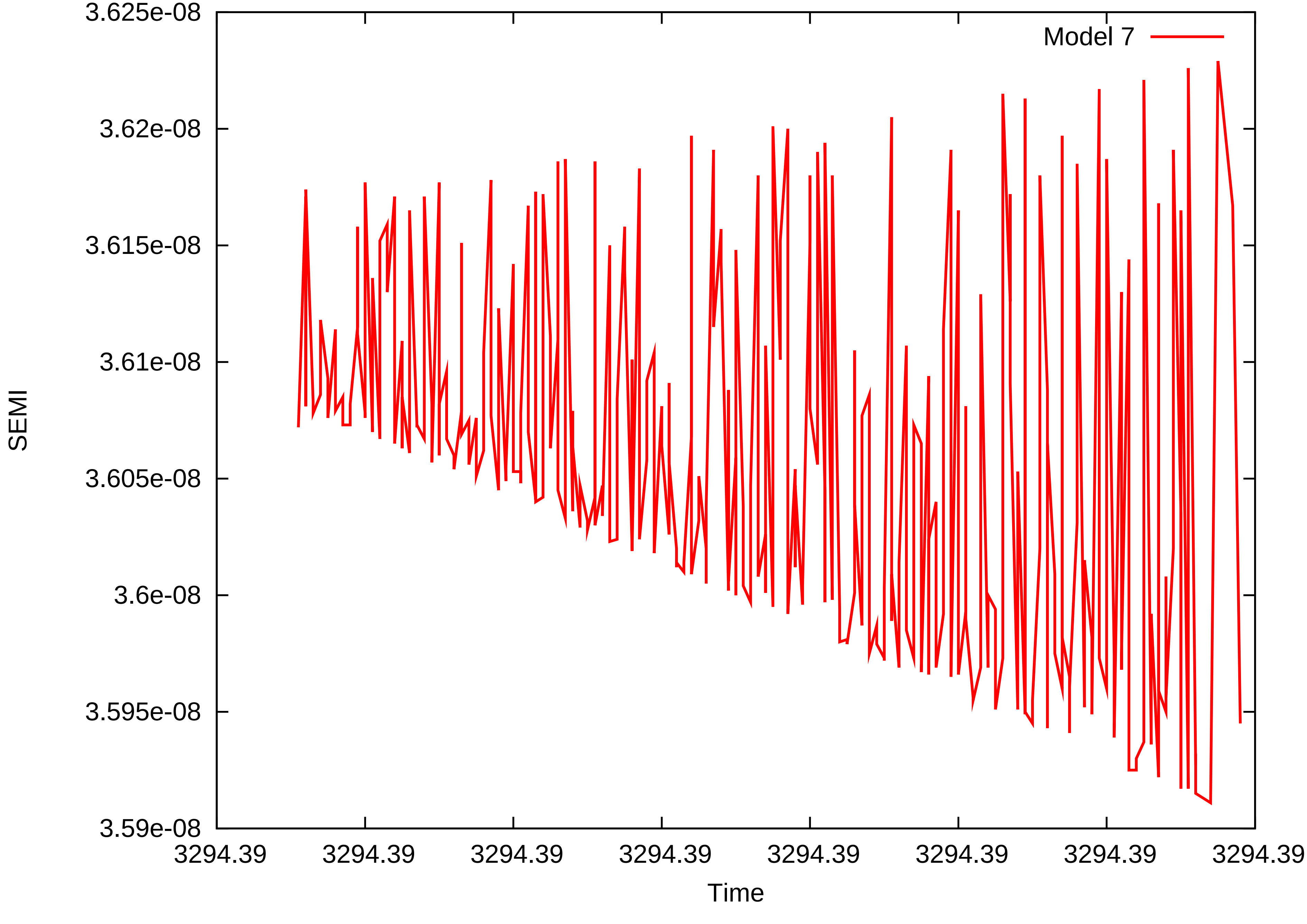


ECC
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
100
200
300
400
500
600
700
800
900
1000

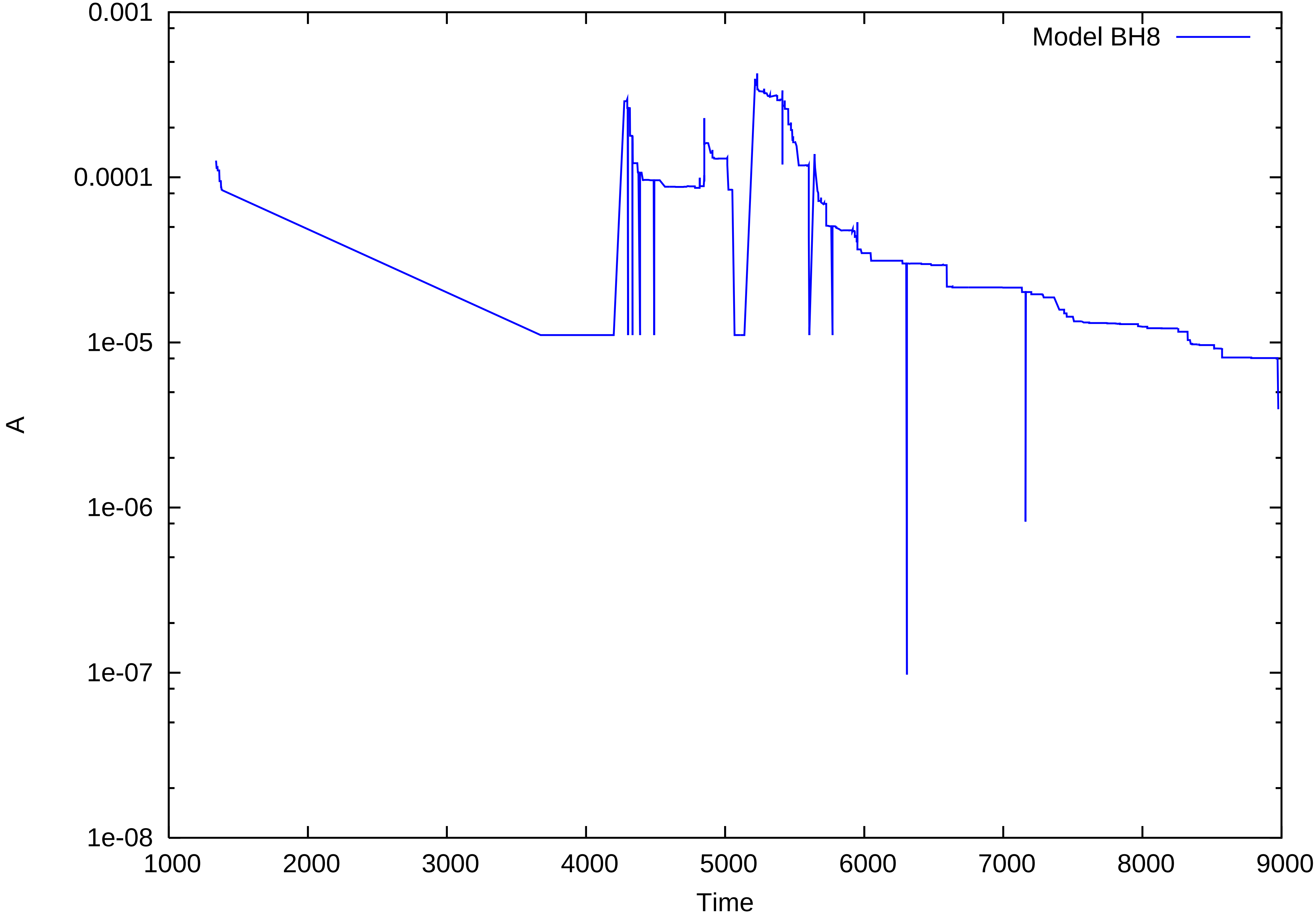
Tidal Disruption

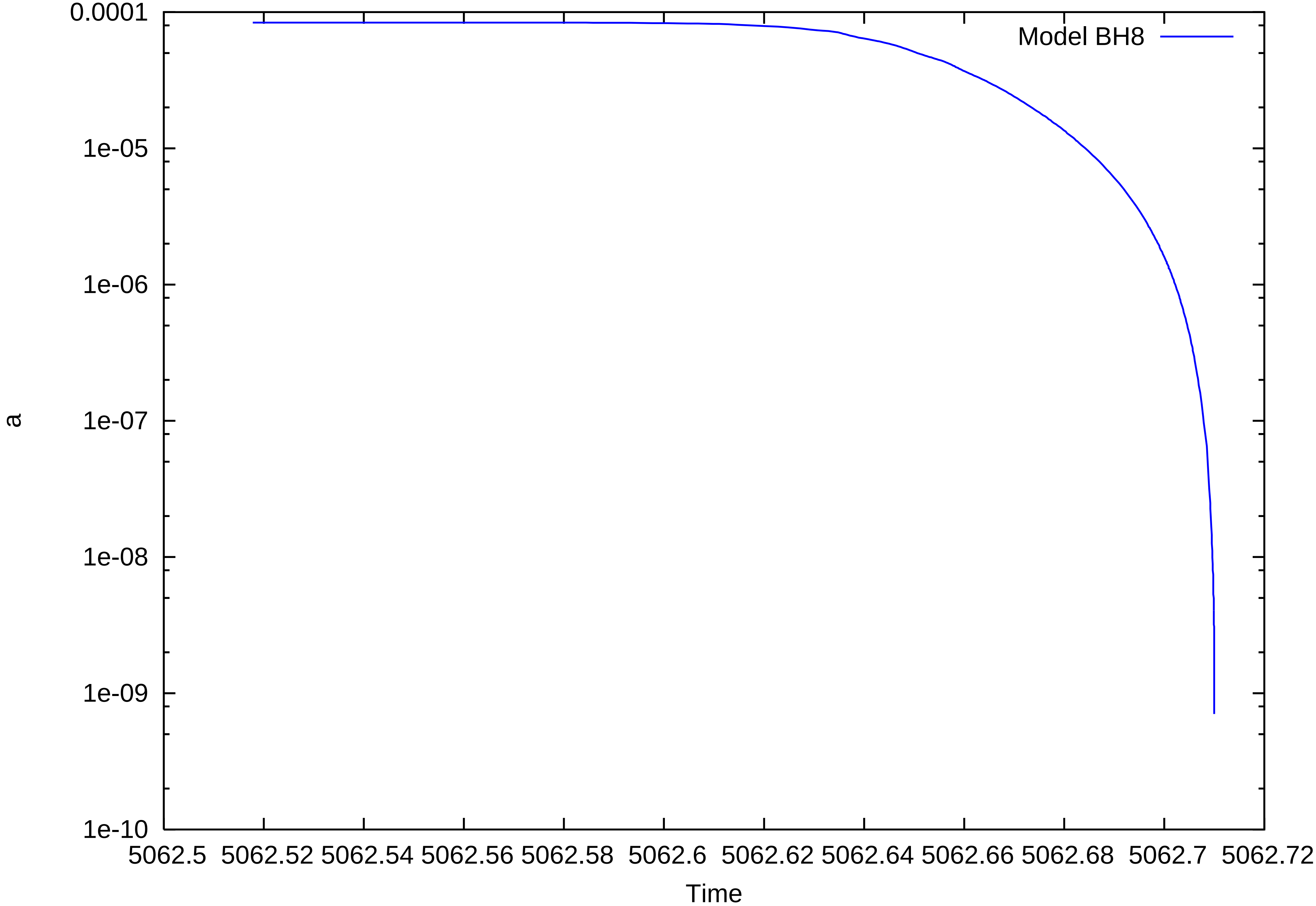
Classical condition	$r_t = \left(\frac{m_{\rm BH}}{m^*}\right)^{1/3} r^*$
Case A	Two-body regularization
	Pericentre test: $a\,(1-e) \ < \ r_t$
	Transform to apocentre $\theta = \pi/2$
	Mass transfer $\Delta m = 0.1\, m^*$
	Escape velocity kick
Corrections	KS elements and energy
Case B	Chain regularization
Mass transfer	$\Delta m = 0.1\, m^*$
Ejection	Energy corrections
Updating	Chain membership and tables
Initialization	Chain c.m. polynomials
Requirements	Large eccentricity
Effects	Mass segregation & Kozai-Lidov
Special cases	WD & AGB cores
Astrophysics	Instantaneous accretion
Observations	Light flickering & emission lines


Model 7
0.378
0.377
0.376
0.375
0.374
0.373
0.372
0.371
0.37
0.369
0.368
0.367
ECC
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
Time

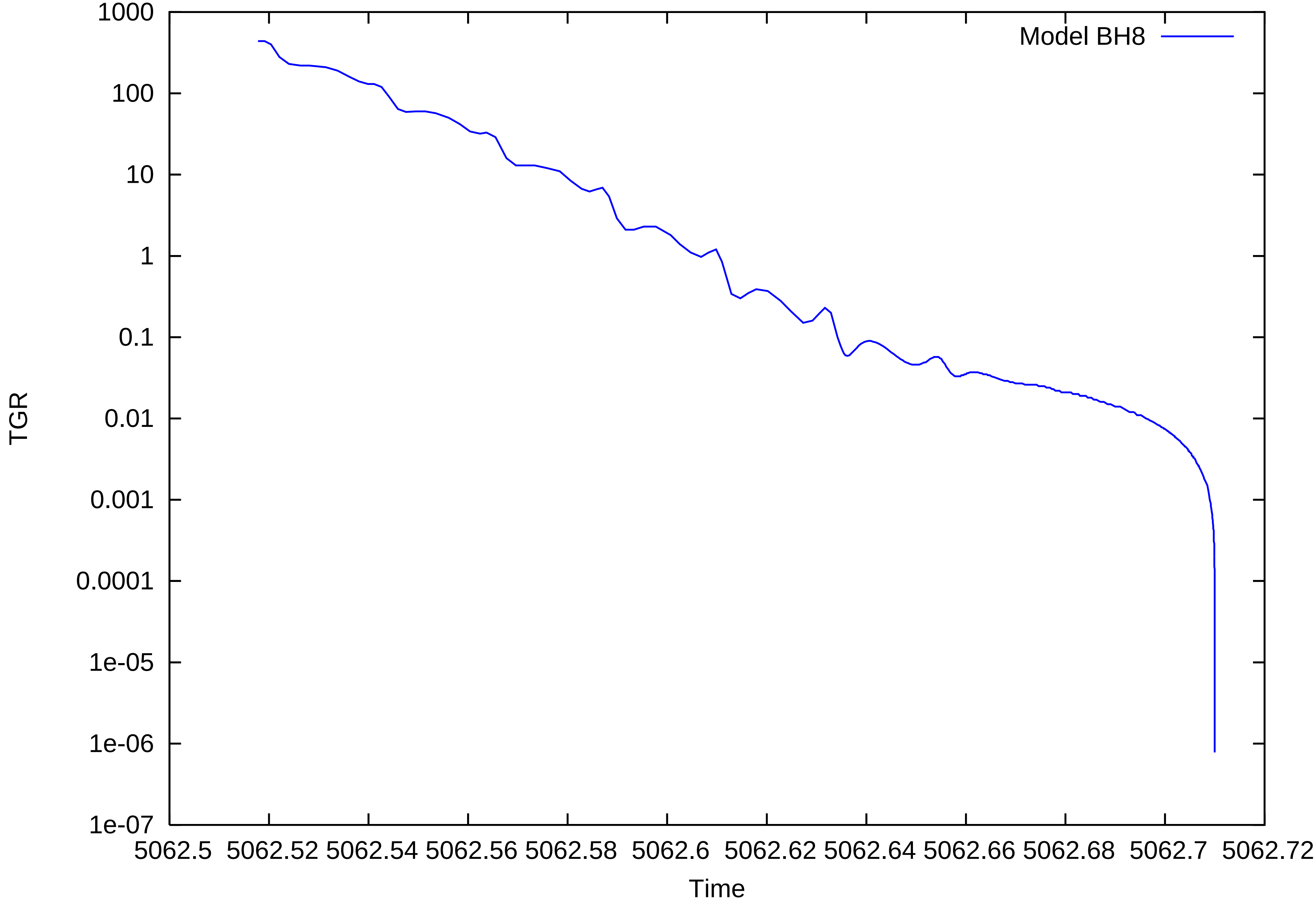


Model 7
3.625e-08
3.62e-08
3.615e-08
3.61e-08
3.605e-08
3.6e-08
3.595e-08
3.59e-08
SEMI
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
3294.39
Time


Model BH8
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1000
2000
3000
4000
5000
6000
7000
8000
9000
Time
A



Model BH8
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
a
5062.5
5062.52
5062.54
5062.56
5062.58
5062.6
5062.62
5062.64
5062.66
5062.68
5062.7
5062.72
Time


Model BH8
TGR
Time
1000
100
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
5062.5
5062.52
5062.54
5062.56
5062.58
5062.6
5062.62
5062.64
5062.66
5062.68
5062.7
5062.72

Discussion

IMF	upper limit, Z
Mass outcome	BH from BSE
PN simulations	speed of light $c = \dfrac{3 \times 10^5}{V^*}$
Dynamics	binary shrinkage & Kozai-Lidov
Slingshot	fast escapers
Retention	velocity kicks
Depletion	ejections of BH & BBH
Observations	BH + S binaries, hyper-velocities
Astropysics	tidal disruption & BH accretion