June 5, 2020

Nonlinear Dynamics, old script 4.2, 4.3

- Methods to check numerical solutions
- Start simple to explain new concepts

Program: Population Dynamics (4.2)
Lorenz Attractor (4.3)

Topics: Fixed Points (FP), Stability, (Quasi)-Periodic Solutions, Deterministic Chaos

4.2. Population Dynamics

Start with very simple ordinary diff. eq.:

Malthus (1798)

$$\frac{dN}{dt} = (b-d)N = rN$$

N: Population number
b, d: birth rate, death rate $[1/T]$
r: net growth rate

[Note: radioactive decay similar]

Dimensionless Form:

$$\tau = rt \quad ; \quad n = \frac{N}{N_0}$$

Usually we start at $\tau_0 = 0$; $0 \leq \tau \leq +\infty$

N_0: free choice; $n, N, N_0 > 0$

$$\Rightarrow \frac{dn}{d\tau} = n \iff \underline{n(\tau) = C \exp(\tau)}$$

C: constant, determined by initial condition.

In original form:

$$\underline{N(t) = N_0 \cdot C \cdot \exp(rt)}$$

With $N(0) = N_0 \Rightarrow C = 1$

Solution leads to unlimited growth $(r > 0)$

~~Ma~~ Verhulst (1836): unrealistic,
finite resources limit growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

K: limiting value for N; $\dot{N} > 0 \iff N < K$
$\dot{N} < 0 \iff N > K$

Dimensionless form:

$\tau = rt$; $n = \frac{N}{K} \Rightarrow$

$$\underline{\frac{dn}{d\tau} = n(1-n)} \quad \Big| \quad \text{Nonlinear Diff. Eq.}$$

Definition: Fixed Point (FP) or Stationary Point

is a point $n = n^*$, for which $\frac{dn}{d\tau} = 0$

Conclusion: for $n(0) = n^*$ the solution is

$n(\tau) = n^* = \text{const.}$ (FP!)

FP's: 1 for Malthus: $n = 0$

2 for Verhulst: $n = 0$, $n = 1$ $(N = K)$

Examine Stability of FP

(what happens to solution $n(\tau)$ near the FP?)

General Form of our diff. eq.: $\frac{dn}{d\tau} = f(n)$

Let n^* be a FP, $f(n^*) = 0$

Let $n(\tau) = n^* + \delta n(\tau)$

be a solution near FP: $\delta n / n \ll 1$

$$\frac{dn}{d\tau} = \frac{d}{d\tau}\left(n^* + \delta n(\tau)\right) = \frac{d}{d\tau}\left(\delta n(\tau)\right)$$

$\hookrightarrow = f(n) = f(n^* + \delta n(\tau)) =$ (Taylor-Series) $= f(n^*) + f'(n^*)\,\delta n(\tau) + \mathcal{O}(\delta n^2)$

By definition of FP we have:

$$f(n^*) = 0 \; ; \quad f(n^*) = \left.\frac{dn}{d\tau}\right|_{n=n^*} = 0$$

So we get:

$$\frac{d}{dt}\,\delta n(\tau) = f'(n^*)\cdot \delta n(\tau) + \mathcal{O}(\delta n^2)$$

First Order only solution is:

$$\delta n(\tau) = \delta n(0)\cdot \exp\left(f'(n^*)\cdot\tau\right)$$

- Case 1: $f'(n^*) < 0 \Longrightarrow$

  $\lim_{\tau\to\infty} \delta n(\tau) = 0$ <u>Stable FP!</u>

  A solution $n(\tau) = n^* + \delta n(\tau)$ approaches the FP: $\lim_{\tau\to\infty} n(\tau) = n^*$

- Case 2: $f'(n^*) > 0 \Longrightarrow$ <u>Unstable FP</u>

  $\lim_{\tau\to\infty} \delta n(\tau) = \pm\infty$ ($\pm$ depends on $\delta n(0)$)

  A solution $n(\tau) = n^* + \delta n(\tau)$ diverges from the FP.

  $\lim_{\tau\to\infty} n(\tau) = \pm\infty$

Two Remarks:

- Case 3: $f'(n^*) = 0$ no result!
  Will be discussed later (Lorenz Attractor)
- Our solution has been obtained by first order Taylor series. The result is valid in some neighbourhood of the FP, but may not be globally valid.

In Tutorial you can check it numerically!

Now let us check our examples:

- Malthus: $\frac{dn}{d\tau} = f(n) = n$; $f'(n) = 1 > 0$

  $\Longrightarrow$ FP $n^* = 0$ is unstable!

- Verhulst: $\frac{dn}{d\tau} = f(n) = n(1-n)$; $f'(n) = 1 - 2n$

$\Rightarrow$ FP1: $n^* = 0$ is unstable!

FP 2: $n^* = 1$ is stable! $f'(n^*) = 1 - 2n^* = -1$

Tutorial: numerical or analytical solution of Verhulst equation $\Longrightarrow$ you can see what it means!

Interacting Populations

Volterra-Lotka System (VL)

N_1 prey $\qquad$ N_2 predator

(more complex: modelling of ecological systems)

$$\frac{dN_1}{dt} = N_1 \left(a - bN_2 \right)$$

$$a, b, c, d > 0$$

$$\frac{dN_2}{dt} = N_2 \left(cN_1 - d \right)$$

N_1 without N_2: Malthusian growth with a

N_2 limits growth of N_1, rate b

N_2 without N_1: „radioactive" decay with d
(no prey => predator disappears)

N_1 limits decay of N_2, rate c
(predator needs prey to survive)

Dimensionless form:

$$\tau = at \; ; \quad u_1 = \frac{c}{d} N_1 \; ; \; u_2 = \frac{b}{a} N_2$$

$$\alpha = \frac{d}{a}$$

$$\frac{du_1}{d\tau} = u_1 (1 - u_2)$$
$$\frac{du_2}{d\tau} = \alpha u_2 (u_1 - 1) \qquad (VL)$$

Only one free parameter α

Notice similarity of 1^{st} eq. to Verhulst eq.

Find the Fixed Points (FP):

FP 1: $u_1^* = u_2^* = 0$

FP 2: $u_1^* = u_2^* = 1$

(Condition for FP in this case:

u_1^*, u_2^* is FP of VL system if

$\frac{du_1}{d\tau} = 0$ and $\frac{du_2}{d\tau} = 0$ at $\begin{array}{l} u_1 = u_1^* \\ u_2 = u_2^* \end{array}$)

Next: Stability Analysis for VL system; it is 2D system.