Lorenz System - Mathematical Description

Very short summary – see other links for details

Density $\rho(x, z, t)$ Gravity $F = (0, 0, -g)$

Pressure $p(x, z, t)$

Temperature $T(x, z, t)$

Velocities $u_x(x, z, t) = u$; $u_z(x, z, t) = w$

Basic equations of hydrodynamics:

- Continuity – mass equation
- Navier Stokes – momentum ; viscosity parameter η
- Heat Transport – energy ; conductivity parameter χ

Approximations:

$$\rho = \rho_0 (1 - \alpha (T - T_0)) \quad \text{linear } \rho\text{-}T\text{-relation}$$

Flow Potential Function $\psi(x, z, t)$ with

$$u = -\frac{\partial \psi}{\partial z} \; ; \; w = \frac{\partial \psi}{\partial x}$$

Temperature Deviation Function $\Theta(x, z, t)$ with

$$T(x, z, t) = T_0 + \Delta T \left(1 - \frac{z}{h}\right) + \Theta(x, z, t)$$

Fourier Series for ψ, θ,

lowest order ansatz:

$$\psi(x,z,t) = \frac{\chi(1+a^2)\sqrt{2}}{a} X(t) \sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)$$

$$\theta(x,z,t) = \frac{\Delta T}{\pi}\frac{Ra_{cr}}{Ra}\left[\sqrt{2}\, Y(t)\cos\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right) - Z(t)\sin\left(\frac{2\pi}{h}z\right)\right]$$

$$\dot{x} = -\sigma x + \sigma y \qquad \sigma = \frac{\nu}{\chi}$$

$$\dot{y} = rx - y - xz \qquad b = \frac{4}{1+a^2}$$

$$\dot{z} = -bz + xy \qquad r = \frac{Ra}{Ra_{cr}}$$

$$Ra = \frac{\alpha g h^3 \Delta T}{\chi \nu} \qquad Ra_{cr} = \frac{\pi^4(1+a^2)^3}{a^2}$$

Dynamics of the Lorenz System

$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{Z} = -bZ + XY$$

Volume contraction

$$\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial y}\dot{y} + \frac{\partial}{\partial z}\dot{z} = -\sigma - 1 - b < 0$$

Trace of Jacobian

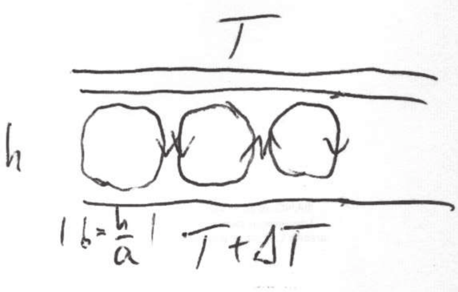
$$\frac{d\log V}{dt} = -\sigma - 1 - b \Rightarrow$$

$$V(t) = V_0 \cdot \exp\{-(\sigma+1+b)\cdot t$$

global attractor!

$\sigma =$ Viscosity / conductivity $\quad b = 4/_{1+a^2}$ geometry

$r = {}^{Ra}/_{Ra_{cr}} \propto \Delta T$



x: velocity, z, y: temperature

Fixed points: $\dot{x} = \dot{y} = \dot{z} = 0$ (4)

(i) $x_1 = y_1 = z_1 = 0$

(ii) $\dot{x} = 0 \Rightarrow x = y$

$\dot{z} = 0 \Rightarrow bz = x^2 \Rightarrow z = \frac{x^2}{b}$

$\dot{y} = 0 \Rightarrow (r-1)x = x^3/b \; (= xz)$

$\Rightarrow x^2 = b(r-1) \; ; \; x_{2/3} = \pm\sqrt{b(r-1)}$

$y_{2/3} = \pm\sqrt{b(r-1)}$

$z_{2/3} = r-1$

Stability of fixed points:

$$J = \begin{pmatrix} -6 & 6 & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

Eigenvalues for $x_1 = y_1 = z_1 = 0$?

$$J := \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\det(J - \lambda E) =$$
$$= (-\sigma-\lambda)(-1-\lambda)(-b-\lambda) - \sigma r(-b-\lambda)$$
$$= -(b+\lambda)\left\{\left[\lambda^2 + (1+\sigma)\lambda + \sigma\right] - \sigma r\right\}$$
$$= -(b+\lambda)\left(\lambda^2 + (1+\sigma)\lambda + \sigma(1-r)\right)$$

$$\lambda_{1/2} = -\frac{\sigma+1}{2} \pm \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(r-1)}$$

$$\lambda_3 = -b$$

$\longrightarrow$

für $r < 1$: $\lambda_{1,2,3} < 0$

$r = 1$: $\lambda_1 = 0$, $\lambda_2 = -(\sigma+1)$, $\lambda_3 = -b$

$r = 0$: $\lambda_{1/2} = -\frac{\sigma+1}{2} \pm \frac{1}{2}(\sigma-1)$: $\lambda_1 = -1$, $\lambda_2 = -\sigma$, $\lambda_3 = -b$

For $r > 1$: $\lambda_1 > 0$ unstable

Stability for x_2, y_2, z_2:

$$\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{pmatrix}$$

$$P(\lambda) = (-\sigma-\lambda)\left[(1+\lambda)(b+\lambda) + b(r-1)\right]$$
$$-\sigma\left[1\cdot(-b-\lambda) + b(r-1)\right]$$
$$= -(+\sigma+\lambda)\left[\lambda^2 + (b+1)\lambda + b + b(r-1)\right]$$
$$-\sigma\left(-b-\lambda + b(r-1)\right]$$
$$= -\Big(\lambda^3 + \lambda^2(\sigma+b+1) + \lambda\big[\underbrace{\sigma(b+1)+b(r-1)}_{-\sigma+b}\big]$$
$$-\big(+\sigma b(r-1) \underbrace{-\sigma b}_{+\sigma b} + 2\sigma b(r-1)\big)\Big)$$
$$= -\left(\lambda^3 + (1+b+\sigma)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1)\right)$$

For $r > 1$ all coeff's > 0

- Third order polynomial has one real root λ_0; $\lambda_0 < 0$!

- All roots are real < 0 for $1 < r < r_1 = 1.34561\ldots$

$ax^3 + bx^2 + cx + d = 0$
Discriminant
$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$

- For $r > r_1$ we get two complex conjugate roots $\lambda_r + i\lambda_i$, $\lambda_r - i\lambda_i$

If $\Delta > 0$ 3 real roots
If $\Delta < 0$ 1 real, 2 compl. conj.

- Characteristic Polynomials

$$P(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_r - i\lambda_i)(\lambda - \lambda_r + i\lambda_i)$$

$$= (\lambda - \lambda_0)(\lambda^2 - 2\lambda\lambda_r + \lambda_r^2 + \lambda_i^2)$$

$$= \lambda^3 + \lambda^2(-\lambda_0 - 2\lambda_r) + \lambda\left((\lambda_r^2 + \lambda_i^2) + 2\lambda_0\lambda_r\right) - \lambda_0(\lambda_r^2 + \lambda_i^2)$$

$$\Rightarrow \quad 1+b+\sigma = -\lambda_0 - 2\lambda_r$$

$$b(r+\sigma) = \lambda_r^2 + \lambda_i^2 + 2\lambda_0\lambda_r$$

$$2\sigma b(r-1) = -\lambda_0(\lambda_r^2+\lambda_i^2)$$

Transition to instability if λ_r becomes positive; so set $\lambda_r = 0 \Rightarrow$

$$2\sigma \cancel{b}(r-1) = \underbrace{(1+b+\sigma)}_{-\lambda_0} \cancel{b} \underbrace{(r+\sigma)}_{\lambda_i^2}$$

$$2\sigma r - 2\sigma = r + br + \sigma r + \sigma + \sigma b + \sigma^2$$

$$r(\sigma - b - 1) = \sigma(3+b+\sigma)$$

$$r = r_{crit} = \frac{\sigma(3+b+\sigma)}{\sigma-b-1}$$

$\boxed{r = r_{crit}!}$

For $r < r_{crit}$:

Lorenz Dynamics – Bifurcation Diagram

$$FP_1 = (0,0,0) \quad ; \quad FP_{2/3} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$