Fr 14.6.19

Lorenz dynamical system

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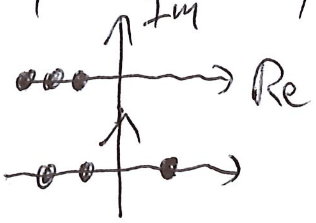
FP_1: $(0,0,0)$; char. Polyn. $-P(\lambda) = (b+\lambda)\left[\lambda^2 + (1+\sigma)\lambda + \sigma(1-r)\right]$

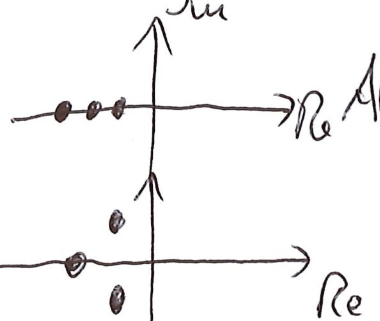


$0 \leq r < 1$ All $\lambda_{1,2,3} < 0$ stable

$r > 1$ One $\lambda_{1,2,3} > 0$ unstable

$FP_{2/3}$: $\left(\pm\sqrt{b(r-1)}; \pm\sqrt{b(r-1)}; r-1\right)$ Only $r>1$

char. Polyn. $-P(\lambda) = \lambda^3 + (1+b+\sigma)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1)$



$1 < r < r_{crit1} = 1.346\ldots$ All $\lambda_{123} < 0$ stable

$r_{crit1} < r < r_{crit2} = 24.74\ldots$ All $Re(\lambda_{123}) < 0$

Two complex conjug. λ stable

$r > r_{crit2}$

$Re(\lambda) > 0$ for comp. conj.

both $FP_{2/3}$ get unstable!

char. Polyn. ansatz: (Theorem of Vieta):

$$-P(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_r - i\lambda_i)(\lambda - \lambda_r + i\lambda_i)$$

with three EW $\lambda_1 = \lambda_0$; $\lambda_2 = \lambda_r + i\lambda_i$; $\lambda_3 = \lambda_r - i\lambda_i$

$$-P(\lambda) = \lambda^3 + \lambda^2(-\lambda_0 - 2\lambda_r) + \lambda\left[\lambda_r^2 + \lambda_i^2 + 2\lambda_0\lambda_r\right] - \lambda_0(\lambda_r^2 + \lambda_i^2)$$

Equate both forms of $P(\lambda)$; comp. coeff:

$$1+b+\sigma = -\lambda_0 - 2\lambda_r$$

$$b(r+\sigma) = \lambda_r^2+\lambda_i^2+2\lambda_0\lambda_r$$

$$2\sigma b(r-1) = -\lambda_0(\lambda_r^2+\lambda_i^2)$$

To find r_{crit2} we look for $\lambda_r = 0$!

$$2\sigma b(r-1) = \underbrace{(1+b+\sigma)}_{-\lambda_0}\underbrace{b(r+\sigma)}_{\lambda_i^2}$$

$$2\sigma r - 2\sigma = r + br + \sigma r + \sigma + b\sigma + \sigma^2$$

$$r(\sigma-b-1) = \sigma(3+b+\sigma)$$

$$r = r_{crit2} = \frac{\sigma(3+b+\sigma)}{\sigma-b-1} \sim 24.74\ldots$$

For $r < r_{crit2}$ $FP_{2/3}$ stable

$r > r_{crit2}$ $FP_{2/3}$ unstable

Volume contraction:

$$\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial y}\dot{y} + \frac{\partial}{\partial z}\dot{z} = -\sigma-1-b < 0$$

(Trace of Jacobian)

Globally contracting! div (velocity) < 0
$\dot{x},\dot{y},\dot{z}$

Definition of the „Attractor" A
(trajectory $\vec{x}(t)$ approaches A and remains on it)

- if $\vec{x}_0 = \vec{x}(t_0) \in A$, $\vec{x}(t) \in A$ for all $t > t_0$
- (in some open environment of A if $\vec{x}_0 = \vec{x}(t_0)$ starts), $\lim_{t \to \infty} \vec{x}(t) \in A$

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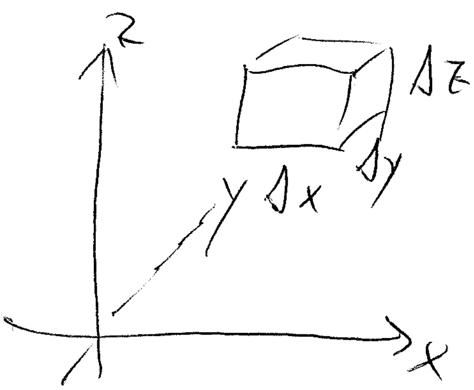
stable FP and limit cycles are attractors

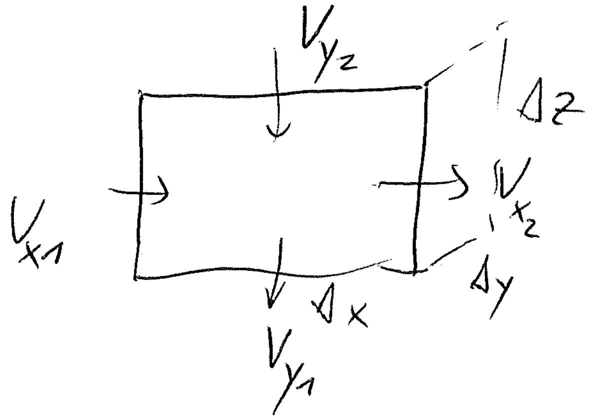
Wed 19.6.19 Lorenz dynamical system

Final Topics: (1) • Contraction



$$V = \Delta x \Delta y \Delta z$$

What happens to volume under Lorenz dynamical equation? (Every point integrated forward in time ...)



$$\Delta V = \underbrace{(V_{x2} - V_{x1})}_{\Delta V_x} \Delta t \Delta y \Delta z + \underbrace{(V_{y2} - V_{y1})}_{\Delta V_y} \Delta t \Delta x \Delta z + \underbrace{(V_{z2} - V_{z1})}_{\Delta V_z} \Delta t \Delta y \Delta x$$

Divide by $\Delta x \Delta y \Delta z \Delta t$, use $\Delta \log V = \frac{\Delta V}{V}$

$$\frac{\Delta \log V}{\Delta t} = \frac{\Delta v_x}{\Delta x} + \frac{\Delta v_y}{\Delta y} + \frac{\Delta v_z}{\Delta z}$$

$\lim_{\Delta x, \Delta y, \Delta z, \Delta t \to 0}$: $\frac{d \log V}{dt} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$v_x = \dot{x}, v_y = \dot{y}, v_z = \dot{z}$ $\qquad = -\sigma - 1 - b < 0!$

(2) Dimension of the attractor A:

- $\dim A \neq 3$ (volume contraction)
- $\dim A \neq 2$ (curves would intersect, more mathematically: Theorem of Poincaré-Bendixson)
- $\dim A \neq 0$ (no stable FP)
- $\dim A \neq 1$? most difficult to show

could be limit cycle
exclude with theory of discrete maps

Theorem of Poincaré - Bendixson
(1D, 2D)

$\dot{\vec{x}} = \vec{f}(\vec{x})$ $\qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

If trajectory $\vec{x}(t)$ remains within closed bounded region D for all $t \geq 0$:

$\Rightarrow$ $\vec{x}(t)$ is

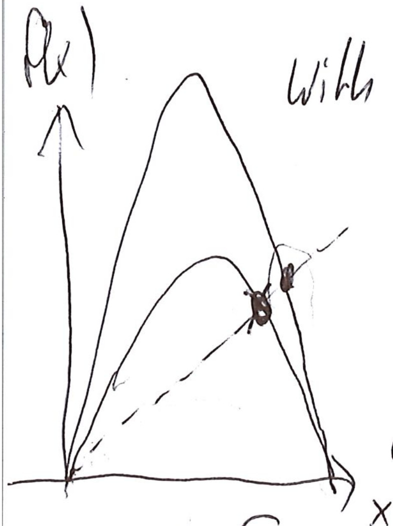
(i) a closed orbit
(ii) approach a periodic orbit (limit cycle)
(iii) approach a fixed point

NB: if there is no stable fixed point, $\vec{x}(t)$ remains in D $\Rightarrow$ limit cycle

(4) Short Excursion Theory of discrete maps

- Set $x_0, x_1, \dots x_n, x_{n+1}, \dots$

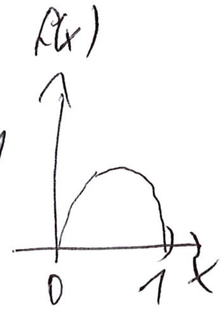
  with $x_{n+1} = f(x_n)$



- Fixed Points and stability

  $x^* = f(x^*)$

- Stability if $|f'(x^*)| < 1$



Example: Logistic Map, $c > 0$

$$x_{n+1} = c x_n \cdot (1 - x_n) = f(x_n)$$

$$(1-2x) \cdot c = c(1-x) - cx = f'(x)$$

Fixed Point: $x = 1 - \frac{1}{c}$, $f'(x) = 2 - c$,

Stable for: $1 < c < 3$

Iterated Map: $x_{n+1} = f^{(p)}(x_n) = f(f(f \dots (f(x_n))$

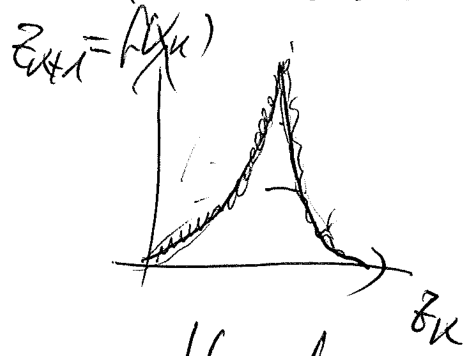
Fixed Point of $f^{(p)}$ is p-periodic point of f

$$\dots \underbrace{x_n, x_{n+1}, \dots x_{n+p-1}}_{x_n = x_{n+p}} \dots$$

Stability of FP of $f^{(p)}$: $|f^{(p)\prime}(x_{n+p})| = |f'(x_{n+p}) \cdots f'(x_n)|$

Also $x_{n+1}, x_{n+2}, \dots$ are FP of $f^{(p)}$, all stable or all unstable

Now look at $z_{k+1} = f(z_k)$ for Lorenz dynamical system as discrete map



Let us assume the Lorenz attractor has a stable p-periodic orbit. Then the discrete map $z_{k+p} = f^{(p)}(z_k)$ must have a stable fixed point.
However, this is impossible, because

$|f'(z_k)| > 1$ everywhere!

And therefore also

$$|f^{p\prime}(z_k)| = \prod_{i=1}^{p} |f'(z_{k+i})| > 1!!$$

Therefore $\dim A \neq 1$!
Caveat: the map (the line) $f(z)$ is not steady! Also fractal ...