8.2. Random Number Transformations

Prepare definition of "probability density function" PDF p(x)

p(x) defines relative likelihood to pick x in a sample

(can be also used to define likelihood of range using integral)

Go back to simple initial examples:

(i): LCG $0 \le I_i \le m-1$

(ii): Normalization $0 \le r_i \le 1$; $r_i = \frac{I_i}{(m-1)}$

Probability Density Function $p(x) = 1$

$$\int_0^1 p(x)\,dx = 1$$

(iii) Normalization on other interval

$0 \le x_i \le a$; $x_i = a r_i$

Equal Probability Density Function $p(x) = \frac{1}{a}$

$$\int_0^a p(x)\,dx = 1$$

8.2. Transforming probability distribution (density) function

Let $p(x)$ be PDF ; $dW = p(x)dx$

probability to find $x, x+dx$

Normalization: $1 = \int_{x_{min}}^{x_{max}} p(x)dx = \int_0^1 p(x)dx$

$p(x)$ is a probability density!

Let $x = x(y)$ be a monotonous function.

Ansatz:

$$p(x)dx = p(x(y)) \left|\frac{dx}{dy}\right| dy = f(y)dy$$

Let us start with eq. distr. $p(x) = 1$: $\Rightarrow$

$$f(y) = \left|\frac{dx}{dy}\right|$$; without loss of generality

$\frac{dx}{dy} > 0$: $f(y) = \frac{dx}{dy} \Longrightarrow$

$$F(y) - F(y_0) = \int_{y_0}^{y} \frac{dx}{dy'}dy' = x - x_0$$

$$= \int_{y_0} f(y')dy'$$

$F(y) = F(y_0) + x - x_0$;

$F(y) = x \Rightarrow y(x) = F^{-1}(x)$

F is indefinite integral (Stammfunktion)

F^{-1} is inverse function of F (Umkehrfunktion)

Example: We want exponentially distributed RV's

$$p(x)\,dx = f(y)\,dy = e^{-y}\,dy = \frac{dx}{dy}\,dy$$

$$F(y) = F(y_0) + \int_{y_0}^{y} e^{-y'}\,dy'$$

$$= F(y_0) - \left[e^{-y'}\right]_{y_0}^{y}$$

$$= F(y_0) - e^{-y} + e^{-y_0}$$

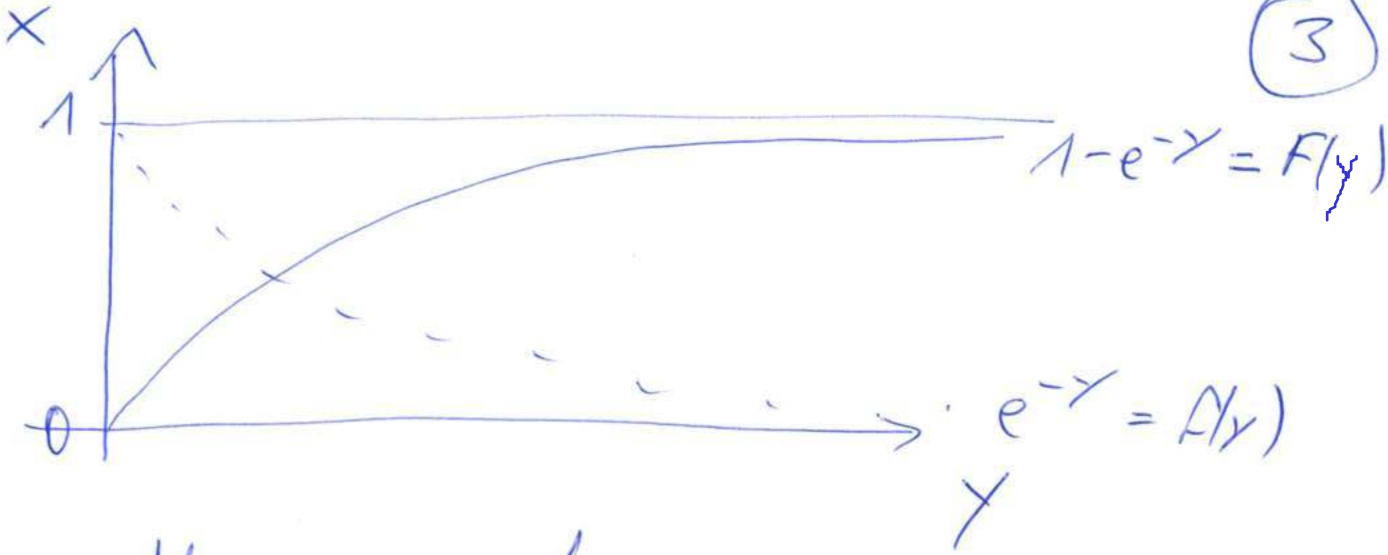
$$= F(y_0) + x - x_0 \qquad \text{Let } x_0 = y_0 = 0$$

$$\Rightarrow e^{-y} = 1 - x$$

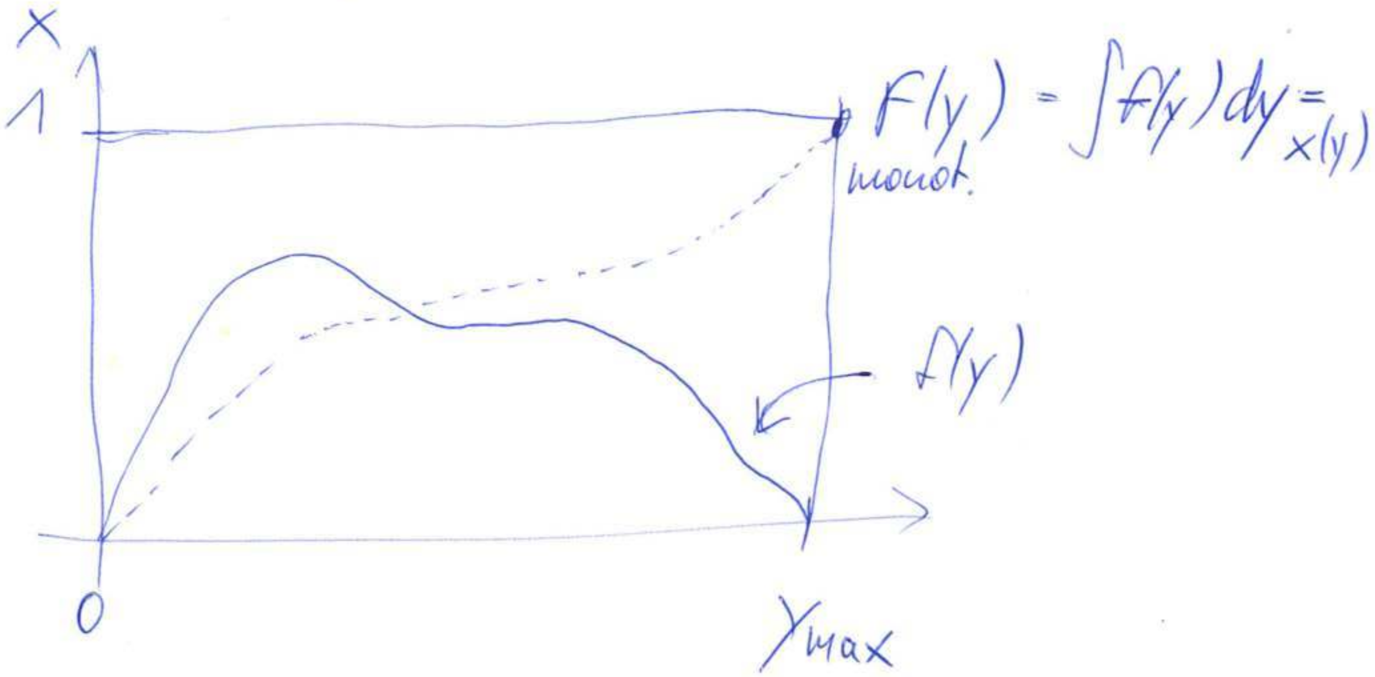
$$y(x) = -\ln(1-x)$$

Note: $x = 1 - e^{-y}$; $\dfrac{dx}{dy} = e^{-y}$

$dx = e^{-y}\,dy$! ✓



More general:



8.2.4. Rejection Method

What if $F^{-1}(y)$ cannot be computed?

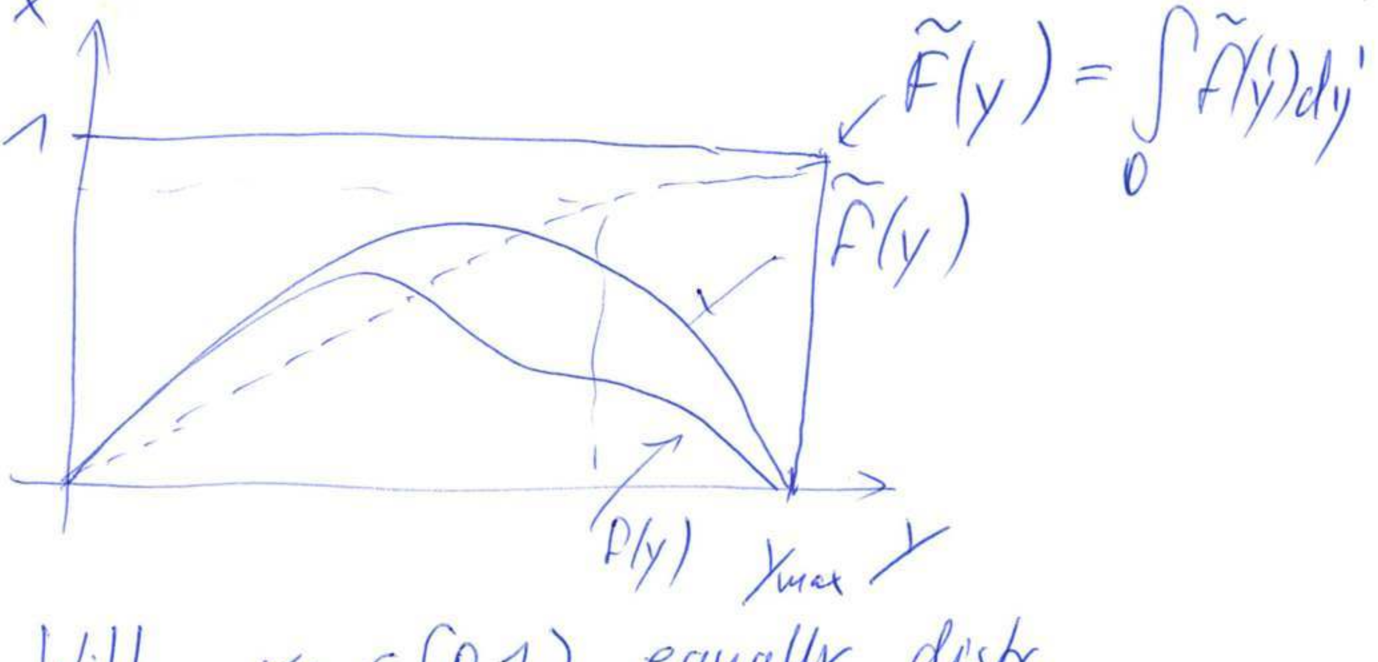
Use majorant $\tilde{f}(y) \geq f(y)$

$(p(x)\,dx = f(y)\,dy)$; then

$$\tilde{F}(y) = \tilde{F}(y_0) + \int_{y_0}^{y} \tilde{f}(y')\,dy' = x - x_0$$

$$x_0 = y_0 = 0 \; ; \; \underbrace{p(x)}_{=1}\,dx = \tilde{f}(y)\,dy$$

$$y = y(x) = \tilde{F}^{-1}(x)$$



With $x_i \in [0,1)$ equally distr.

$y_i = \tilde{F}^{-1}(x_i)$ is distr. acc. to PDF $\tilde{f}(y)$

Next: choose eq. distr. RN $x' \in [0, \tilde{f}(y)]$

If $x' \leq f(y)$ accept. (prob. $\frac{f(y)}{f(y')}$)

If $x' > f(y)$ reject

Seq. $x_0, x_1, \quad x_i, x_{i+1}, \ldots$

$\qquad\quad \uparrow \qquad\qquad \uparrow$

$\qquad$ Rej. $\qquad\quad$ Rej.

Seq. $x_0', \quad x_2', \ldots x_i' \ldots x_{i+2}', \ldots$

$y_i' = \tilde{F}^{-1}(x_i')$ is distr. acc. to $f(y)$!

Good majorant: $\tilde{f}(y) = \dfrac{c_0}{1 + (y - y_m)^2 / a_0^2}$

Maximum is at $y = y_m$; $\tilde{f}(y_m) = c_0$

FWHM $= 2a_0$

$$\tilde{F}(y) = a_0 c_0 \operatorname{arctg}\left(\frac{y - y_m}{a_0}\right) + \tilde{c}$$

Let $x_0 = 0, \ y_0 = 0$

$$x = \tilde{F}(y) = a_0 c_0 \operatorname{arctg}\left(\frac{y - y_m}{a_0}\right) + \tilde{c}$$

$$= \int_0^y \tilde{f}(y')\,dy'$$

$$x_0 = \tilde{F}(y_0) = 0 = a_0 c_0 \operatorname{arctg}\left(\frac{-y_m}{a_0}\right) + \tilde{c}$$

$$\Rightarrow \qquad \tilde{c} = + a_0 c_0 \operatorname{arctg}\frac{y_m}{a_0}$$

$$\Rightarrow \ y(x) = y_m + a_0 \tan\left(\frac{x}{a_0 c_0} - \operatorname{arctg}\frac{y_m}{a_0}\right)$$

$y_i = y(x_i)$ distr. acc. $\tilde{f}(y)$

Box-Muller Algorithm: Gaussian PDF

$$\underset{=1}{p(x_1)}\underset{=1}{p(x_2)}\,dx_1 dx_2 = f(y_1) f(y_2)\, dy_1 dy_2$$

$$= |\det J|\, dy_1\, dy_2$$

$$= \left|\frac{\partial f(y_1, y_2)}{\partial x_1, x_2}\right| dy_1 dy_2$$

Claim: $y_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2)$

$y_2 = \sqrt{-2\ln x_1}\sin(2\pi x_2)$ does it:

$$y_1^2 + y_2^2 = -2\ln x_1 \Rightarrow x_1 = \exp\left(-\tfrac{1}{2}y_1^2 - \tfrac{1}{2}y_2^2\right)$$

$$y_2 / y_1 = \tan(2\pi x_2) \Rightarrow x_2 = \frac{1}{2\pi}\operatorname{arctg}\left(\frac{y_2}{y_1}\right)$$

$$\det J = \frac{1}{2\pi}\left(\frac{-\exp(--)}{\left(1+\frac{y_2^2}{y_1^2}\right)} - \frac{-\frac{y_2^2}{y_1^2}\cdot\exp(--)}{\left(1+\frac{y_2^2}{y_1^2}\right)}\right)$$

$$J = \begin{pmatrix} -y_1 \exp(\ ) & -y_2 \exp(-\ ) \\ \frac{1}{2\pi}\,\frac{\cancel{0} - \frac{y_2}{y_1^2}}{1+\frac{y_2^2}{y_1^2}} & \frac{1}{2\pi}\,\frac{\frac{1}{y_1}}{1+\frac{y_2^2}{y_1^2}} \end{pmatrix}$$

$$= \frac{\exp(-y_1^2/2)}{\sqrt{2\pi}} \cdot \frac{\exp(-y_2^2/2)}{\sqrt{2\pi}}$$