

NBODY6:

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Methods

Implementations

Post-Newtonian Treatment

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# Newton's Equations

Force       $\mathbf{F}_i = -G \sum_{j=1; j \neq i}^N m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$

Explicit differentiation

$$\begin{aligned}\mathbf{F}_i^{(1)} &= -G \sum_{j=1; j \neq i}^N m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \\ &\quad - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}\end{aligned}$$

New solution at  $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = \left( \frac{1}{2} \mathbf{F}_i^{(1)} \Delta t + \mathbf{F}_i \right) \Delta t$$

$$\Delta \mathbf{r}_i = \left( \left( \frac{1}{6} \mathbf{F}_i^{(1)} \Delta t + \frac{1}{2} \mathbf{F}_i \right) \Delta t + \dot{\mathbf{r}}_i \right) \Delta t$$

# Hermite Integration

Taylor series for  $\mathbf{F}$  and  $\mathbf{F}^{(1)}$

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)} t + \frac{1}{2} \mathbf{F}_0^{(2)} t^2 + \frac{1}{6} \mathbf{F}_0^{(3)} t^3$$

$$\mathbf{F}^{(1)} = \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)} t + \frac{1}{2} \mathbf{F}_0^{(3)} t^2$$

Prediction

$$\mathbf{r}_j = ((\frac{1}{6} \mathbf{F}_0^{(1)} \delta t'_j + \frac{1}{2} \mathbf{F}_0) \delta t'_j + \mathbf{v}_0) \delta t'_j + \mathbf{r}_0$$

$$\mathbf{v}_j = ((\frac{1}{2} \mathbf{F}_0^{(1)} \delta t'_j + \mathbf{F}_0) \delta t'_j + \mathbf{v}_0); \quad \delta t'_j = t - t_0$$

New forces  $\mathbf{F}, \mathbf{F}^{(1)}$

Higher derivatives

$$\mathbf{F}_0^{(3)} = (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)}) t) \frac{6}{t^3}$$

$$\mathbf{F}_0^{(2)} = (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)}) t) \frac{2}{t^2}$$

Corrector for  $i$

$$\Delta \mathbf{r}_i = \frac{1}{24} \mathbf{F}_0^{(2)} \Delta t^4 + \frac{1}{120} \mathbf{F}_0^{(3)} \Delta t^5$$

$$\Delta \mathbf{v}_i = \frac{1}{6} \mathbf{F}_0^{(2)} \Delta t^3 + \frac{1}{24} \mathbf{F}_0^{(3)} \Delta t^4$$

# Neighbour Scheme

Total force  $\mathbf{F}(t) = \sum_{j=1}^n \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n \ll \Delta t_d, \quad n \ll N$$

Neighbour sphere  $R_s^{\text{new}} = R_s^{\text{old}} \left( \frac{n_p}{n} \right)^{1/3}$

Neighbour selection  $|\mathbf{r}_i - \mathbf{r}_j| < R_s, \Rightarrow \text{list}$

Derivative corrections  $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

# Basic Regularization

Two-body equations     $\ddot{x} = -\frac{M}{x^2}$      $h = \frac{1}{2}\dot{x}^2 - \frac{M}{x}$

Time smoothing                   $t' \equiv \frac{dt}{d\tau} = x; \quad \Rightarrow \frac{d}{dt} = \frac{1}{x} \frac{d}{d\tau}$

Using     $\ddot{x} = -\frac{x'^2}{x^3} + \frac{x''}{x^2}$      $x'' = \frac{x'^2}{x} - M$

Substitution     $\dot{x} = \frac{x'}{x}$      $x'' = 2hx + M$

Coordinate transformation     $u^2 = x$

Twice differentiation                   $u'' = -\frac{u'^2}{u} + \frac{hx}{u} + \frac{M}{2u}$

Simplifying     $\frac{u'^2}{u} = \frac{1}{4}u\dot{x}^2$      $u'' = \frac{1}{2}hu$

Regular equation for  $x \Rightarrow 0$

# Code Structure

Input	Read input parameters
Initial conditions	Generate & scale $m, \mathbf{r}, \dot{\mathbf{r}}$
Initialization	$\mathbf{F}, \mathbf{F}^{(1)}$ & $\Delta t$
Scheduling	Form block-step distribution
New time	$T_{\text{block}} = \min_j(t_j + \Delta t_j)$
Close encounters	Two-body regularization
Prediction	Neighbours or all $N$
Neighbour integration	Sequential $\mathbf{F}_n \& \mathbf{F}_n^{(1)}$
Regular force calc	Sequential total force
Stellar evolution	Updating mass & radius

## Units

(a) Scaling relations

Given length scale  $R_V$  in pc and total mass  $N M_S$  in  $M_\odot$

Velocity scaling

$$\tilde{V}^* = 1 \times 10^{-5} (GM_\odot/L^*)^{1/2} \text{ km/s, with } L^* = 3 \times 10^{18} \text{ cm}$$

Velocity unit       $V^* = 6.557 \times 10^{-2} (NM_S/R_V)^{1/2} \text{ km/s}$

Fiducial time       $\tilde{T}^* = (L^{*3}/GM_\odot)^{1/2} = 14.94 \text{ Myr}$

Time unit       $T^* = 14.94 (R_V^3/NM_S)^{1/2} \text{ Myr}$

(b) Conversion from N-body to physical units

$$\begin{aligned} \tilde{r} &= R_V r \text{ pc, } \tilde{v} = V^* v \text{ km/s, } \tilde{t} = T^* t \text{ Myr,} \\ \tilde{m} &= NM_S m \text{ } M_\odot \end{aligned}$$

Crossing time     $T_{\text{cr}} = 2\sqrt{2} T^* \text{ Myr}$

# Scaling of Initial Conditions

Main input	$N, N_{\text{b}}, M_{\text{S}}, R_{\text{pc}}$
Cluster parameters	optional IMF and Plummer or King model
Initial data	$\tilde{m}_i, \tilde{\mathbf{r}}_i, \tilde{\mathbf{v}}_i, \dots, i = 1, N$
Total energy	$E = T - U$
Virial theorem	$\mathbf{v}_i = q \tilde{\mathbf{v}}_i, q = \left[ \frac{Q_{\text{V}} U}{T} \right]^{1/2}, \mathbf{r}_i = \tilde{\mathbf{r}}_i$
Standard units	$G = 1, \Sigma m_i = 1, E_0 = -0.25$
Standard scaling	$\hat{\mathbf{r}}_i = \frac{\mathbf{r}_i}{S^{1/2}}, \hat{\mathbf{v}}_i = \mathbf{v}_i S^{1/2}, S = \frac{E_0}{q^2 T - U}$
Astrophysical units	$V^*, T^*, R^*$ from $M_{\text{tot}}, R_{\text{pc}}$
Primordial binaries	split or copy $m_i$ , introduce $a, e, \Omega$
Force polynomials	$\mathbf{F}_i, \dot{\mathbf{F}}_i, \Delta t_i, \dots, i = 1, N$
KS regularization	explicit initialization, $R < R_{\text{cl}}$

# Time-Steps

Basic time-step       $\Delta t = \frac{\alpha |\mathbf{r}|}{|\mathbf{v}|}, \quad \Delta t = \frac{\beta |\mathbf{F}|}{|\mathbf{F}^{(1)}|}$

Taylor series     $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)} \Delta t + \frac{1}{2} \mathbf{F}_0^{(2)} \Delta t^2 + \dots$

Natural time-step     $\Delta t = \left( \frac{\eta |\mathbf{F}|}{|\mathbf{F}^{(2)}|} \right)^{1/2}, \quad \eta = 0.02$

General expression     $\Delta t = \left( \frac{\eta (|\mathbf{F}| |\mathbf{F}^{(2)}| + |\mathbf{F}^{(1)}|^2)}{|\mathbf{F}^{(1)}| |\mathbf{F}^{(3)}| + |\mathbf{F}^{(2)}|^2} \right)^{1/2}$

Relative criterion     $\Delta t$  independent of mass

Block-steps     $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Hierarchical levels     $\mathcal{N}_k$  particles with steps  $\Delta t_k$

Scheduling             $i = \min (t_j + \Delta t_j)$

# Softening

Potential  $\Phi(r) = \frac{Gm}{(r^2 + \epsilon^2)^{1/2}}$

Energy  $E = \frac{1}{2}v^2 - \Phi(r) = \text{const}$

Force  $\mathbf{F}_i = -G \sum_{j=1; j \neq i}^N m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$

Force derivative

$$\begin{aligned}\mathbf{F}_i^{(1)} &= -G \sum_{j=1; j \neq i}^N m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}} \\ &\quad - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}\end{aligned}$$