Exercises Lecture Computational Physics (Summer 2010)
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Week 5 (Whitsun - Week)

(This is an extra sheet outside of the normal series)

Three-Body Problems - Preparation

We will study the dynamics of three masses m_1, m_2, m_3, with initial positions $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, with $\mathbf{x}_i = (x_i, y_i, z_i)$ $(i = 1, 2, 3)$ in Cartesian coordinates; and velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ $(\mathbf{v}_i = (v_{xi}, v_{yi}, v_{zi}))$.

First write down Newton's equations of motion in standard vector form (second order diff. eq. using only positions and their second derivative). Second, transform these equations to a system of first order differential equations by using $\dot{\mathbf{x}}_i = \mathbf{v}_i$ and $\ddot{\mathbf{x}}_i = \dot{\mathbf{v}}_i$ (the dot denotes here as usual in physics the independent variable t for time).

Be aware that in total we have now a system of 18 coupled first order differential equations, if written down coordinate by coordinate. To simplify the problem we will only assume planar problems, i.e. the motion of our particles happens only in the x, y-plane. ($z_i = v_{zi} = 0$). Also you can assume a scaling where the gravitational constant $G = 1$. Now we have 12 differential equations left.

Third rewrite your differential equations in mathematical variables y, with y_i, $(i = 1, \ldots 12)$, such that y_{1+4i}, y_{2+4i} $(i = 0, 1, 2)$ are the coordinates and $y_{3+4i}, y_{4+4i} = \dot{y}_{1+4i}, \dot{y}_{2+4i}$ are the velocities of the bodies. Use your Runge-Kutta 4 program to solve this system of differential equations by integrating forward in time. You may keep t as an independent variable; so we are now looking for 12 unknown functions $y_i(t)$ of our differential equations. For the practical exercises below we will give initial data in the y_i notation.

2. The three-body choreography

Use the following starting conditions for your RK4 solver:

$$
\begin{aligned}
(y_1, y_2) &= -0.97000436 \quad 0.24308753 \\
(y_3, y_4) &= -0.46620368 \quad -0.43236573 \\
(y_5, y_6) &= 0.97000436 \quad -0.24308753 \\
(y_7, y_8) &= -0.46620368 \quad -0.43236573 \\
(y_9, y_{10}) &= 0.0 \quad 0.0 \\
(y_{11}, y_{12}) &= 0.93240737 \quad 0.86473146
\end{aligned}
$$

Choose masses $m_1 = m_2 = m_3 = 1$ and a time step h between 0.01 und 0.001.
Questions:

- Plot the solution (trajectories of three bodies in their plane of motion). 5 points.
- Perturb the solution by changing one of the initial conditions or one of the masses by 0.01. Integrate three of the perturbed solutions for long time (1000 time units). Plot the solution and describe in words what happens. What do you think about the stability of the solution? 5 points
- Try to increase the perturbation until the solution completely unstable (three body system decays). Plot one unstable solution and describe the final state in this case in words. 5 points

3. Burrau's Three Body Problem

Use the following initial conditions:

$$
\begin{aligned}
(y_1, y_2) &= 0.0 \quad 0.0 \\
(y_3, y_4) &= 0.0 \quad 0.0 \\
(y_5, y_6) &= 3.0 \quad 0.0 \\
(y_7, y_8) &= 0.0 \quad 0.0 \\
(y_9, y_{10}) &= 0.0 \quad 4.0 \\
(y_{11}, y_{12}) &= 0.0 \quad 0.0
\end{aligned}
$$

Choose masses $m_1 = 5$, $m_2 = 4$, $m_3 = 3$ and a time step h between 0.01 und 0.001. This is called Burrau's Three-Body Problem, and is known as one of the most difficult and unstable three-body problems. The initial positions of three masses at rest are in a rectangular triangle with side lengths $3, 4, 5$, where the body of mass 5 is opposite to the side with length 5, the body of mass 4 is opposite to the side with length 4, and so on.

Integrate the problem for 100 time units, and write down a list of times and minimum distances of closest pairwise encounters between any two of the three particles. Redo this with smaller and smaller step size h. Do you reach convergence for the list of minimum distances? 5 points.