Exercises Lecture Computational Physics (Summer 2011)

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1 Heat conduction equation

Consider an electricity cable consisting of a metal core of radius $R_{\rm core} = 0.1\,\rm cm$ surrounded by an insulating plastic mantel of 0.1 cm thickness, making the entire cable $R_{\rm tot} = 0.2\,\rm cm$ in radius. The metal core produces heat as a result of the electricity flowing through it. The heat production rate in units of Watt per $\rm m^3$ is q, which we assume to be a constant inside the metal core, but zero in the insulating mantel.

$$q(R) = \begin{cases} 10^8 & \text{for} \quad R \le R_{\rm core} \\ 0 & \text{for} \quad R > R_{\rm core} \end{cases} \tag{1}$$

The heat conduction flux is given by the following equation:

$$j(R) = -\kappa\,\frac{\mathrm{d}T(R)}{\mathrm{d}R} \tag{2}$$

where $T(R)$ is the temperature in Kelvin at radius R, and κ is the conductivity coefficient which we assume to be constant here. We take $\kappa = 1\,\rm W\,m^{-1}\,K^{-1}$ for the moment.

Now let us discretize this on an R-grid with:

$$R_i = (i - 1/2) \times \Delta R \tag{3}$$

for some given constant grid spacing ΔR. The cell centers are at $i = 1 \ldots N$, the cell walls are at $i = 1/2 \ldots N + 1/2$. This gives N cells and $N + 1$ walls (the first wall being at $R = 0$).

- Assuming that you want the last cell wall $R_{N+1/2}$ to be exactly at $R_{\rm tot}$, and given some integer value of N, how should you choose ΔR?
- Write down the analytical form of the *steady state* conservation equation[1] for the flux j, with source function q.
- Show that this can be discretized as

$$\frac{1}{R_i\,\Delta R}\left\{\kappa_{i+1/2}R_{i+1/2}\frac{T_i - T_{i+1}}{\Delta R} - \kappa_{i-1/2}R_{i-1/2}\frac{T_{i-1} - T_i}{\Delta R}\right\} = q_i \tag{4}$$

- You can form this into a tridiagonal matrix equation, and solve this using the subroutine you programmed last week. Write out explicit expressions for the a_i, b_i, c_i and r_i (where small-r stands for "right-hand-side" while big-R is the coordinate) for $i = 2 \ldots N - 1$.

[1]Remember that in cylindrical coordinates, with symmetry in ϕ and z, the divergence of a flux $\vec{j}$ is $\nabla \cdot \vec{j} = R^{-1}\,\mathrm{d}(R\,j_R)/\mathrm{d}R$

2 Heat conduction equation continued (Homework)

We now must impose boundary conditions. At $R = 0$ we wish to impose

$$\frac{\mathrm{d}T(R)}{\mathrm{d}R} = 0 \qquad \text{at} \quad R = 0 \tag{5}$$

while at the outer edge we set

$$T(R) = 20 \qquad \text{at} \quad R = R_{\mathrm{tot}} \tag{6}$$

1. (5 pt) How can you write the discretized versions of these boundary conditions?
2. (5 pt) How does this translate into the matrix elements b_1, c_1, a_N, b_N and right-hand-sides r_1 and r_N?
3. (5 pt) Write a computer program, using the tridiagonal matrix equation solver you programmed last week[2], that solves for T_i for $i = 1 \ldots N$. Plot the results.
4. (5 pt) Now make κ non-constant: Take $\kappa = 10$ for $R \leq R_{\mathrm{core}}$ and $\kappa = 1$ for $R > R_{\mathrm{core}}$. What changes?

[2] You can also use the `tridag()` subroutine from Numerical Recipes, if you prefer that.