8 Random Numbers

Principle: Linear congruential generators:

$$I_{n+1} = (a\, I_n + c) \bmod m$$

General Formula:

$$I_n = (a^n I_0 + c_n) \bmod m$$

with $c_n = c \cdot \sum_{i=0}^{n-1} a^i$

Proof by Induction: $n = 1$ ok

$$I_{n+1} = (a\,[a^n I_0 + c_n] + c) \bmod m$$

$$= (a^{n+1} I_0 + a c_n + c) \bmod m$$

To show: $c_{n+1} = a c_n + c$

$$c_{n+1} = c \cdot \sum_{i=0}^{n} a^i = a c \sum_{i=0}^{n-1} a^i + c \quad \surd$$

Max period m ; discrete map

Nec. + suff. condition for max period:

- no common prime factors betw. c and m
  $m = m_0 m_1$, $c = c_0 m_1$ $\quad c \bmod m = c_0 m_1 \bmod m_0 m_1$
- $a-1$ div. by all prime factors of m

Get $a^n \bmod m$ cover all $1, \dots, m-1$

Claim: The Liapounov coefficient of discrete map $I_{n+1} = (a I_n + c) \bmod m$ is $\log a > 0 \Rightarrow$ one condition for chaos, but periodic!

$$\delta_n = I_n - I_n' = \left(a^n I_0 + c_n - a^n I_0' - c_n\right) \bmod m$$

$$= a^n (I_0 - I_0') \bmod m = a^n \delta_0 \bmod m$$

Assume $n <$ Period of LCG $\leq m$

$$\delta_n = \delta_0 \exp(n \lambda_n) = \delta_0 a^n \Rightarrow$$

$$n \lambda_n = n \log a \quad \surd$$

Probability Distribution Function $p(x)$

Random Numbers $0 \leq I_i \leq m-1$

Normalize 1: $0 \leq r_i \leq a$ by

$$r_i = I_i / (m-1)$$

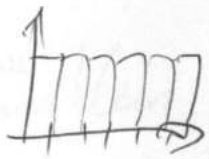
Normalize 2:

$$1 = \int_0^a p(x)\,dx = \sum x_i \Delta x_i \; ; \quad \text{often } a = 1!$$

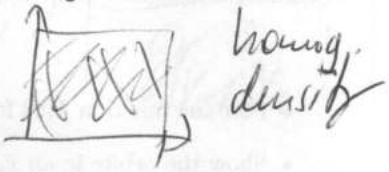
LCG and also ran2.c give equally distributed RN's

- How we can see this?

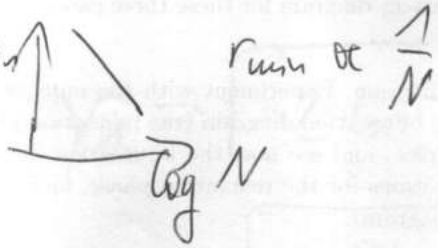
1. Histogram, bins



2. Two sequences plot against each other



3. Smallest RN r_{min}



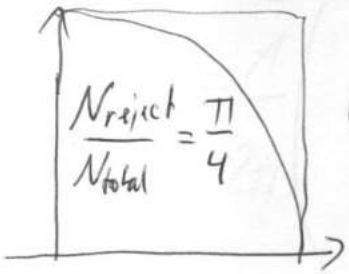
How to get other distributions?

- Rejection Method

(i) random number r_i, $x_i = a r_i \in [0, a]$

(ii) 2nd random number s_i, if $s_i < f(x_i)$ take
if $s_i > f(x_i)$ reject

- Compute π by RN: $r_i \in [0,1)$



2nd RN: s_i, $s_i < \sqrt{1 - r_i^2}$ take
$s_i > \sqrt{1 - r_i^2}$ reject