Final Lecture July 22, 2020

- Ising Model – additional informations
- Lorenz Dynamics – " "
- Summary of Lecture

Spin Reversal in Ising Model

$S \to S'$; only one j with $s_j' = -s_j$

for all $\alpha \neq j$: $s_\alpha' = s_\alpha$

$$\Delta E = \Delta H = H(S') - H(S)$$

$$= -b \sum_\alpha (s_\alpha' - s_\alpha) - j \sum_{\langle \alpha\beta \rangle} (s_\alpha' s_\beta' - s_\alpha s_\beta)$$

$\Delta s_j = s_j' - s_j$

$$= -b \, \Delta s_j - j \sum_\beta \Delta s_j \, s_\beta - j \sum_{\substack{\langle\alpha\rangle \\ \text{nb of } j}} s_\alpha \, \Delta s_j$$

$$= -\Delta s_j \left(b + 2j \sum_{\substack{\beta \\ \text{nb of } j}} s_\beta \right)$$

$$\text{max/min} = \pm(2b \pm 16 j) \quad \left(\text{for } \sum s_\beta = \pm 4\right)$$

$$\Delta s_j = \pm 2 \; ; \quad \sum_{\beta (\text{nb of } j)} s_\beta = +4, +2, 0, -2, -4$$

Possible other values only:

$$\Delta E = \pm (2b \pm 8j) \qquad \left(\sum s_\beta = \pm 2\right)$$

$$\pm (2b) \qquad \left(\sum s_\beta = \pm 0\right)$$

In total only 5 possible values of ΔE.

- Lorenz Dynamics – additional informations

For $r > r_{crit} = 24\ldots$ what is the property of the attractor? A

- $\dim A \neq 0$ (no stable FP)
- $\dim A \neq 1$ to be shown (no limit cycle)
- $\dim A \neq 2$ curves would intersect
  more mathematically:
  Theorem of Poincaré–Bendixson
- $\dim A \neq 3$ volume contraction of Lorenz dynamical equations

Theorem of Poincaré Bendixson

Let D be closed bounded region in (x,y) plane, and

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

be a dynamical system (f,g cont. diff'ble)

If a trajectory is such that it remains in D for all $t \geq t_0$, then the trajectory must be

(i) a closed (periodic) orbit (limit cycle)

(ii) approach a closed (per.) orbit

(iii) approach a fixed point

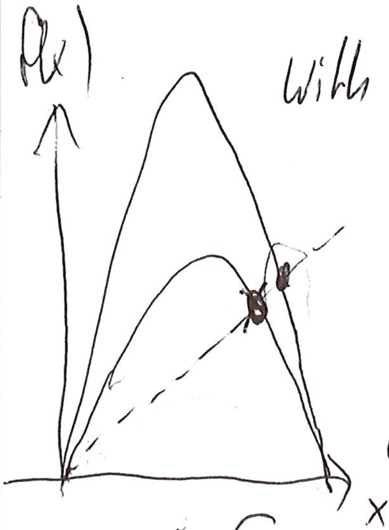
as $t \to \infty$

Consequence: if no FP in D, and trajectory remains in D

$\rightarrow$ limit cycle

(4) Short Excursion Theory of discrete maps

- ~~Punkte~~ Set $x_0, x_1, \dots x_n, x_{n+1}, \dots$

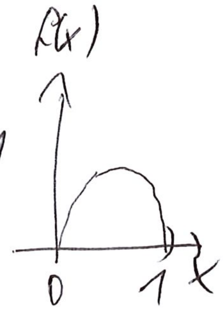


with $x_{n+1} = f(x_n)$

- Fixed Points and stability

$$x^* = f(x^*)$$

- Stability if $|f'(x^*)| < 1$



Example: Logistic Map, $c > 0$

$$x_{n+1} = c x_n \cdot (1 - x_n) = f(x_n)$$

$$(1-2x) \cdot c = c(1-x) - cx = f'(x)$$

Fixed Point: $x = 1 - \frac{1}{c}$, $f'(x) = 2 - c$

Stable for: $1 < c < 3$

Iterated Map: $x_{n+1} = f^{(p)}(x_n) = f(f(f \dots (f(x_n)))$

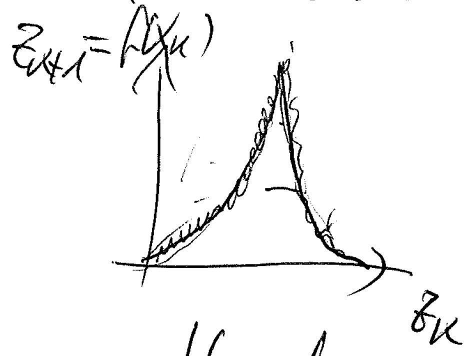
Fixed Point of $f^{(p)}$ is p-periodic point of f

$$\dots \underbrace{x_n, x_{n+1}, \dots x_{n+p-1}}_{x_n = x_{n+p}} \dots$$

Stability of FP of $f^{(p)}$: $|f^{(p)\prime}(x_{n+p})| = |f'(x_{n+p}) \cdots f'(x_n)|$

Also $x_{n+1}, x_{n+2}, \dots$ are FP of $f^{(p)}$, all stable or all unstable

Now look at $z_{k+1} = f(z_k)$ for Lorenz dynamical system as discrete map



Let us assume the Lorenz attractor has a stable p-periodic orbit. Then the discrete map $z_{k+p} = f^{(p)}(z_k)$ must have a stable fixed point. However, this is impossible, because $|f'(z_k)| > 1$ everywhere!

And therefore also

$$|f^{p\,\prime}(z_k)| = \prod_{i=1}^{p} |f'(z_{k+i})| > 1 !!$$

Therefore $\dim A \neq 1$!

Caveat: the map (the line) $f(z)$ is not steady! Also fractal ...