1 D Regularization

mu = m1 m2 / (m1 + m2)reduced

1. Coordinate Transformation $r = u^2$ r = L(u)u

$$H = \frac{p^2}{2\mu} - \frac{GMm}{r} = E_0 = \text{const.} \qquad p = \mu \dot{r} = 2u\dot{u}\mu$$

Canonical Trafo: $p\dot{r} = P\dot{u} \Rightarrow P = 4u^2\dot{u}\mu$

$$H = \frac{P^2}{8u^2\mu} - \frac{GMm}{u^2} = \text{const.}\ E_0$$

Pdot = dH/duudot

P = 4 u' mu

2. Time Transformation

$$dt = r\,ds = u^2 ds \; ; \qquad \dot{u} = \frac{du}{dt} = \frac{1}{r}\frac{du}{ds} \Rightarrow$$

$$= g(p, r)$$
$$= g(P, u)$$

ds/dt = 1/r

$$u^2\dot{u} = u' = \frac{1}{4}\frac{P}{\mu}$$

u' = du/ds

3. Poincare-Transform:

$$0 = \Gamma = g(P,u)\, H(P,u) = \frac{P^2}{8\mu} - GMm - E_0 u^2$$

4. Canonical Eq: $P' = \frac{\partial \Gamma}{\partial u} = -2E_0 u = 4u''\mu$

$$\Rightarrow u'' + \frac{1}{2}\frac{E_0}{\mu} u = 0$$ harmonic oscillator (if $E_0 < 0$)

$$\omega^2 = \frac{E_0}{2\mu}$$ half frequency

2D

Need

$$r = L(u)u = u \circ u = u^2 = \left[\begin{pmatrix}u_1\\u_2\end{pmatrix}\right]^2$$

$$\dot{r} = 2u\dot{u}$$

Vector!

Complex numbers

Levi-Civita

(u1 + i u2)**2

= u1**2 - u2**2 + 2u1 u2 i

u1(u1**2 - u2**2(

u**2 = L(u) u

u1   -u2L(u)

Levi-Civita Transformation 2-D

$$H = \mu \frac{V^2}{2} - \frac{G m_1 m_2}{R} = H(V, R)$$

$$(u \cdot u)^2 = R \iff L(u)u = R \iff L(u) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$$

$$2L(u)\dot{u} = \dot{R} = V$$

$$\left.\begin{aligned} &4u^2\dot{u}^2 = \dot{R}^2 = V^2 . \\ \Rightarrow \quad &P = 4u^2\dot{u} = 2L^T(u) \cdot V \end{aligned}\right\} \Rightarrow V^2 = \frac{P^2}{4u^2}$$

$$\Rightarrow \quad \frac{1}{2u^2} L(u) \cdot P = V$$

$$H = \mu \frac{P^2}{8u^2} - \frac{G m_1 m_2}{u^2} = H(P, u)$$

Time Transformation

$$\frac{dt}{ds} = R = u^2$$

Poincaré Transform

$$\Gamma = R(H - E) = \frac{\mu P^2}{8} - Eu^2 - G m_1 m_2$$

harmonic oscillator $\omega^2 = \frac{\mu}{4E}$

half frequency

9.6.99 (1)

Are our transformations canonical?

(Levi-Civita plus time transformation)

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$ S: action integral, L: Lagrange function

$\delta S = 0$ defines physical motion (least action principle, Hamilton's variational princ.)

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$ = (partial integr.) (Variat. of boundary terms vanishes)

$$= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt$$

For all δq_i $\Rightarrow$ Euler-Lagrange Eq. of motion!

$$H = \sum_i p_i \dot{q}_i - L \iff L = \sum_i p_i \dot{q}_i - H$$

$$0 = \delta S = \int_{t_1}^{t_2} \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right) dt$$

$$= \int_{t_1}^{t_2} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] dt$$

For all $\delta q_i, \delta p_i$ $\Rightarrow$ Hamilton's Eq. of motion.

Canonical Transformation $\Leftrightarrow$
$\Leftrightarrow$ Hamilton's Eq. of motion invariant

$$\delta \int_{t_1}^{t_2} \left(\sum p_i \dot{q}_i - H \right) dt = 0$$ in both systems of variables.

1) Levi-Civita transformation

$$\sum_{i=1}^{3} p_i \dot{q}_i = \sum_{i=1}^{3} v_i \dot{r}_i = \sum_{i=1}^{3} v_i^2$$ $p_i = v_i$, $q_i = r_i$

Let $Q_i = u_i$, $P_i = 4u^2 \dot{u}_i$ Levi-Civita variables

$$\sum_{i=1}^{4} P_i \dot{Q}_i = 4u^2 \sum_{i=1}^{4} \dot{u}_i^2 = \sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{3} p_i \dot{q}_i$$

since $\sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{4} 4u^2 \dot{u}_i^2$ see before

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2) Time transformation $dt = g(Q, t)\, ds$

$$\delta S = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{3} p_i \dot{q}_i - H(p,q) \right) dt = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{4} P_i \dot{Q}_i - H(P,Q) \right) dt$$

$$= \delta \int_{s_1}^{s_2} \left(\frac{1}{g} \sum_{i=1}^{4} P_i Q_i' - H(P,Q) \right) g\, ds$$

$$\boxed{Q_i' = \frac{\partial Q_i}{\partial s} = \frac{\partial Q_i}{\partial t} \cdot \frac{\partial t}{\partial s} = \dot{Q}_i \cdot g \, ; \; \Gamma = g\left(H - \frac{h}{\mu}\right)} = \delta \int_{s_1}^{s_2} \left(\sum_{i=1}^{4} P_i Q_i' - \Gamma \right) ds$$

$p_i, q_i, H(p_i, q_i, t), t$ satisfies Hamilton's eq.

$$\Longleftrightarrow$$

$P_i, Q_i, \Gamma(P_i, Q_i, s), s$ satisfies Hamilton's eq.

Γ: Poincaré transform of Hamiltonian

$$\Gamma = g\left(H - \frac{h}{\mu}\right) \qquad dt = g \cdot ds$$

Our example: $\quad g = r = u^2 = \sum_{i=1}^{4} u_i^2$

$$H = \frac{1}{2}\sum_{i=1}^{3} v_i^2 - \frac{Gm_1m_2}{\mu r} = 2u^2 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu u^2}$$

$$\Gamma = r \cdot \left(H - \frac{h}{\mu}\right) = 2u^4 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu} - \frac{h}{\mu}u^2$$

$$= \sum_{i=1}^{4}\left(\frac{P_i^2}{8} - \frac{h}{\mu}Q_i^2\right) - \frac{Gm_1m_2}{\mu}$$

Canonical Equations:

$$Q_i' = \frac{\partial \Gamma}{\partial P_i} = \frac{P_i}{4} \Longleftrightarrow u_i' = u^2 \dot{u}_i = r\dot{u}_i \quad \text{O.K.}$$

$$P_i' = -\frac{\partial \Gamma}{\partial Q_i} = 2\frac{h}{\mu}Q_i \Longleftrightarrow \left(4u^2\dot{u}_i\right)' = 4u_i'' = 2\frac{h}{\mu}u_i$$

$u_i'' - \frac{h}{2\mu}u_i = 0$	$u_i'' + \frac{\lvert h\rvert}{2\mu}u_i = 0 \quad (h<0)$

Henceforth:

Time transformation $dt = R_1 R_2 \, ds$

Poincaré Hamiltonian $\Gamma = R_1 R_2 \left(H - \quad \right)$

e.g. $\vec{R}_1 = \vec{r}_1 - \vec{r}_3$
$\vec{R}_2 = \vec{r}_2 - \vec{r}_3$ for three bodies!

Hamiltonian equations $\Longrightarrow$

regularized equations of motion.

They are not regular!

Next: Aarseth + Zare '74:
3-body regularization.

Three-Body Regularization

i) Three-Body Hamiltonian

$$H = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2\right)$$

$$- \frac{G m_1 m_2}{r_1 - r_2} - \frac{G m_2 m_3}{r_2 - r_3} - \frac{G m_1 m_3}{r_1 - r_3}$$

$$\vec{V}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3} = \frac{\sum m_i \vec{v}_i}{M}$$

$$\vec{V}_1 = \vec{v}_1 - \vec{v}_3 \qquad \vec{R}_1 = \vec{r}_1 - \vec{r}_3$$

$$\vec{V}_2 = \vec{v}_2 - \vec{v}_3 \qquad \vec{R}_2 = \vec{r}_2 - \vec{r}_3$$

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \qquad r_i = |\vec{r}_i| \quad v_i = |\vec{v}_i|$$

$$R_i = |\vec{R}_i| \quad V_i = |\vec{V}_i|$$

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 \qquad (1)$$

$$m_3\vec{V}_1 = m_3\vec{v}_1 - m_3\vec{v}_3 \qquad (2)$$

$$m_3\vec{V}_2 = m_3\vec{v}_2 - m_3\vec{v}_3 \qquad (3)$$

$$M\vec{v}_{cm} + m_3\vec{V}_1 = (m_1+m_3)\vec{v}_1 + m_2\vec{v}_2 \qquad (4)=(1)+(2)$$

$$M\vec{v}_{cm} + m_3\vec{V}_2 = (m_2+m_3)\vec{v}_2 + m_1\vec{v}_1 \qquad (5)=(1)+(3)$$

$$(m_2+m_3)M\vec{v}_{cm} + (m_2+m_3)m_3\vec{V}_1 = (m_2+m_3)(m_1+m_3)\vec{v}_1 + (m_2+m_3)m_2\vec{v}_2$$

$(4) * (m_2+m_3) = (6)$

$$-m_2M\vec{v}_{cm} - m_2m_3\vec{V}_2 = -m_1m_2\vec{v}_1 - m_2(m_2+m_3)\vec{v}_2$$

$(5) * (-m_2) = (7)$

$$m_3M\vec{v}_{cm} + (m_2+m_3)m_3\vec{V}_1 - m_2m_3\vec{V}_2 =$$

$$(8) \qquad = (m_2+m_3)(m_1+m_3)\vec{v}_1 - m_1m_2\vec{v}_1$$

$$= \left(m_2m_3 + m_3(m_1+m_3)\right)\vec{v}_1 = m_3M\vec{v}_1$$

$$\vec{v}_1 = \vec{v}_{CM} + \frac{m_2+m_3}{M}\vec{V}_1 - \frac{m_2}{M}\vec{V}_2 \qquad (8)/(m_3M)$$

$$\vec{v}_2 = \vec{v}_{CM} + \frac{m_1+m_3}{M}\vec{V}_2 - \frac{m_1}{M}\vec{V}_1 \qquad (9)$$ analogous 1 and 2 exchanged

$$\vec{V}_3 = \left(M\vec{V}_{cm} - m_1\vec{v}_1 - m_2\vec{v}_2 \right) / m_3 =$$

$$= \Big(M\vec{V}_{cm} - m_1\vec{v}_{cm} - \frac{m_1}{M}(m_2+m_3)\vec{V}_1 + \frac{m_1 m_2}{M}\vec{V}_2$$
$$- m_2\vec{v}_{cm} - \frac{m_2}{M}(m_1+m_3)\vec{V}_2 + \frac{m_1 m_2}{M}\vec{V}_1 \Big) / m_3$$

$$= \vec{V}_{cm} - \frac{m_1}{M}\vec{V}_1 - \frac{m_2}{M}\vec{V}_2 \qquad (10)$$

Transformation:

$$T = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2 \right) = \frac{1}{2}\Big(M v_{cm}^2 + m_1 \frac{(m_2+m_3)^2}{M^2} V_1^2 + \frac{m_1 m_2^2}{M^2} V_2^2$$

$$+ m_2 \frac{(m_1+m_3)^2}{M^2} V_2^2 + \frac{m_2 m_1^2}{M^2} V_1^2 + \frac{m_3 m_1^2}{M^2} V_1^2 + \frac{m_3 m_2^2}{M^2} V_2^2$$

$$- 2\frac{m_1 m_2(m_2+m_3)}{M^2} V_1 V_2 - 2\frac{m_1 m_2(m_1+m_3)}{M^2} V_1 V_2 + 2\frac{m_1 m_2 m_3}{M^2} V_1 V_2$$

$$+ 2\frac{m_1(m_2+m_3)}{M} V_1 v_{cm} + 2\frac{m_2(m_1+m_3)}{M} V_2 v_{cm} - 2\frac{m_1 m_3}{M} V_1 v_{cm}$$

$$- 2\frac{m_1 m_2}{M} V_2 v_{cm} - 2\frac{m_1 m_2}{M} V_1 v_{cm} - 2\frac{m_2 m_3}{M} V_2 v_{cm} \Big)$$

Collect Terms: $V_1 V_2$:

$$\frac{2}{M^2}\left(m_1 m_2 m_3 - m_1 m_2 (m_1+m_3) - m_1 m_2 (m_2+m_3) \right) =$$

$$= \frac{2}{M^2}\left(-m_1 m_2 m_3 - m_1^2 m_2 - m_1 m_2^2 \right) = -\frac{2 m_1 m_2}{M}$$

Collect Terms V_1^2:

$$\frac{1}{M^2}\left(m_1(m_2+m_3)^2 + m_2 m_1^2 + m_3 m_1^2\right) =$$

$$= \frac{1}{M^2}\left(m_1 m_2^2 + 2m_1 m_2 m_3 + m_1 m_3^2 + m_2 m_1^2 + m_3 m_1^2\right)$$

$$= \frac{1}{M^2} m_1 \left(m_2^2 + m_3^2 + 2m_2 m_3 + m_1(m_2+m_3)\right)$$

$$= \frac{1}{M^2} m_1 (m_2+m_3)(m_1+m_2+m_3) = \frac{m_1(m_2+m_3)}{M}$$

Analogous V_2^2: $\frac{m_2(m_1+m_3)}{M}$

All Terms $V_1 v_{cm}$, $V_2 v_{cm}$ vanish! $\Longrightarrow$

$$H = \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\frac{m_1(m_2+m_3)}{M} V_1^2 + \frac{1}{2}\frac{m_2(m_1+m_3)}{M} V_2^2$$

$$- \frac{m_1 m_2}{M} V_1 V_2 - \frac{G m_1 m_2}{R_1 - R_2} - \frac{G m_2 m_3}{R_2} - \frac{G m_1 m_3}{R_1}$$

Note: $r_1 - r_2 = R_1 - R_2$ has been used.

Note: $V_1 - V_2 = V_{rel}$, $R_1 - R_2 = R_{rel}$ $\Longrightarrow$

$$H = \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\frac{m_1 m_2}{M} V_{rel}^2 - \frac{G m_1 m_2}{R_{rel}}$$

$$+ \frac{1}{2}\frac{m_1 m_3}{M}\left(V_1^2 \underset{\wedge\ \frac{m_2 m_3}{M}}{+} V_2^2\right) - \frac{G m_2 m_3}{R_2} - \frac{G m_1 m_3}{R_1}$$

Regularization:

$dt = R_1 R_2 \, ds$ $\qquad \vec{Q}_i, \vec{P}_i \in \mathbb{R}^4$

$R_1 = Q_1^2 \qquad \vec{R}_1 = L(\vec{Q}_1)\vec{Q}_1$

$R_2 = Q_2^2 \qquad \vec{R}_2 = L(\vec{Q}_2)\vec{Q}_2$

$\vec{P}_1 = 4Q_1^2 \dot{\vec{Q}}_1 = 4\vec{Q}_1' / Q_2^2 \qquad \vec{V}_1 = 2 \cdot L(\vec{Q}_1) \frac{\vec{P}_1}{4Q_1^2}$

$\vec{P}_2 = 4Q_2^2 \dot{\vec{Q}}_2 = 4\vec{Q}_2' / Q_1^2 \qquad \vec{V}_2 = 2 \cdot L(\vec{Q}_2) \frac{\vec{P}_2}{4Q_2^2}$

$$V_1^2 = \dot{R}_1^2 = 4Q_1^2 \dot{Q}_1^2 = \frac{P_1^2}{4Q_1^2}; \quad V_2^2 = \frac{P_2^2}{4Q_2^2}$$

$$\Gamma = R_1 R_2 (H - E) = Q_1^2 Q_2^2 (H - E)$$

$$= \frac{m_1(m_2+m_3)}{M} \frac{P_1^2 Q_2^2}{8} + \frac{m_2(m_1+m_3)}{M} \frac{P_2^2 Q_1^2}{8}$$

$$- \frac{m_1 m_2}{M} \cdot \frac{L(\vec{Q}_1)\vec{P}_1 \, L(\vec{Q}_2)\vec{P}_2}{4}$$

$$- \frac{G m_1 m_2 Q_1^2 Q_2^2}{|\vec{R}_1 - \vec{R}_2|} - G m_2 m_3 Q_1^2 - G m_1 m_3 Q_2^2 - E Q_1^2 Q_2^2$$

Next: Equations of Motion!

Three body singularity is essential!

3-6 Equations of Motion

30.6.99 (1)

$$\vec{Q}_1' = \frac{\partial \Gamma}{\partial \vec{P}_1} = \frac{m_1(m_2+m_3)}{M}\frac{P_1 Q_2^2}{4} - \frac{m_1 m_2}{M}\frac{L^T(\vec{Q}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$\vec{P}_1' = -\frac{\partial \Gamma}{\partial \vec{Q}_1} = \frac{m_2(m_1+m_3)}{M}\frac{\vec{Q}_1 P_2^2}{4} - \frac{m_1 m_2}{M}\frac{L^T(\vec{P}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$- \frac{2Gm_1m_2}{|R_1-R_2|}\vec{Q}_1 Q_2^2 - 2Gm_2m_3\vec{Q}_1 - 2E\vec{Q}_1 Q_2^2$$

$$\vec{Q}_1' = \frac{\vec{P}_1}{4} \qquad \vec{P}_1' = 4\vec{Q}_1'' = \vec{Q}_1\vec{X} + \vec{Y}$$

$$\vec{X} = \frac{m_2(m_1+m_3)}{M}\frac{P_2^2}{4} - \frac{2Gm_1m_2}{|\vec{R}_1-\vec{R}_2|}Q_2^2 - 2EQ_2^2$$

$$\vec{Y} = -\frac{m_1m_2}{M}\frac{L^T(\vec{P}_1)L(\vec{Q}_2)\vec{P}_2}{4}$$

$$\boxed{\vec{Q}_1'' - \omega^2\vec{Q}_1 = \frac{1}{4}\vec{Y}} \qquad \omega^2 = \frac{1}{4}\vec{X}$$

Harmonic Oscillator with external quasi-periodic triggering $\Rightarrow$ resonances, strongly chaotic system (double pendulum!)

More Regularizations?

- 3-body

Aarseth 85: $$dt = \frac{R_1 R_2}{\sqrt{R_1 + R_2}}\, ds$$

- N-body

Heggie 74: $$dt = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} R_{ij}\, ds$$

$$R_{ij} = |\vec{r}_i - \vec{r}_j| = L(\vec{Q}_{ij})\,\vec{Q}_{ij}$$

$$\vec{V}_{ij} = \dot{\vec{R}}_{ij} = 2L(\vec{Q}_{ij})\,\vec{P}_{ij} / 4Q_{ij}^2$$

Mikkola 97

$$H(\vec{V}_{ij}, \vec{R}_{ij}) = \sum_{i<j} \frac{V_{ij}^2}{2\mu_{ij}} + \ldots + \sum_{i<j} \frac{G m_i m_j}{R_{ij}}$$

$$\Gamma = \left(H(V_{ij}, R_{ij}) - E \right) \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} R_{ij}$$

$$= \left[H\left(\frac{L(Q_{ij})P_{ij}}{2Q_{ij}^2}, L(Q_{ij})Q_{ij} \right) - E \right] \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} Q_{ij}^2$$

Heggie's Global Regularization

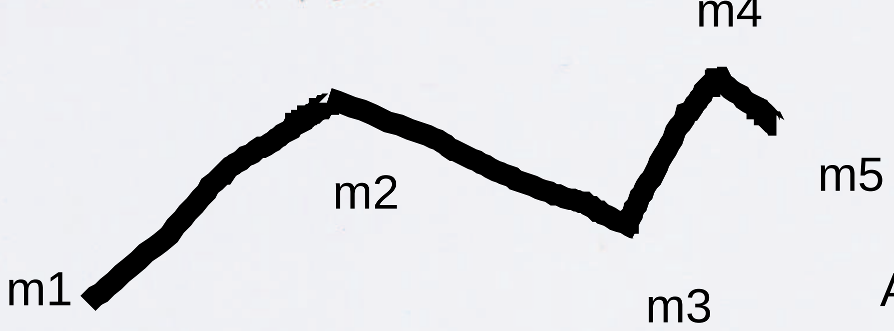
Very Complicated Equations of Motion!

High Dimensionality: $N(N-1)/2$

- Chain Method (Mikkola)

$$\vec{R}_k = \vec{r}_{k+1} - \vec{r}_k \qquad \vec{R}_k = L(\vec{Q}_k)\vec{Q}_k$$

$$dt = \frac{1}{T+U}\, ds$$



Algorithmic Regulariza

- Slow-Down Treatment Mikkola, Aarseth 96

$$\ddot{\vec{r}} = -\frac{G m_1 m_2}{\kappa^2 |\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) + \vec{F}_{ext}$$

Slow-Down coefficient κ

$$\dot{\vec{r}} = \frac{1}{\kappa} \cdot \vec{v}$$

Period is κ-fold longer!

- Stumpff-Functions

$$\vec{u}'' + \frac{|h|}{2\mu} \vec{u} = \frac{u^2}{2} L^T(\vec{u}) \vec{F}_{ext} = \frac{u^2}{2}$$

Use Stumpff functions for series evaluation of solution. See later...