9.6.99 (1)

Are our transformations canonical?

(Levi-Civita plus time transformation)

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$ S: action integral
L: Lagrange function

$\delta S = 0$ defines physical motion
(least action principle, Hamilton's variational princ.)

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$ = (partial integr.)
(variat. of boundary terms vanishes)

$$= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt$$

For all $\delta q_i \Rightarrow$ Euler-Lagrange Eq. of motion!

$$H = \sum_i p_i \dot{q}_i - L \iff L = \sum_i p_i \dot{q}_i - H$$

$$0 = \delta S = \int_{t_1}^{t_2} \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right) dt$$

$$= \int_{t_1}^{t_2} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] dt$$

For all $\delta q_i, \delta p_i \Rightarrow$ Hamilton's Eq. of motion.

Canonical Transformation $\Longleftrightarrow$

$\Longleftrightarrow$ Hamilton's Eq. of motion invariant

$$\delta \int_{t_1}^{t_2} \left(\sum p_i \dot{q}_i - H \right) dt = 0$$ in both systems of variables.

1) Levi-Civita transformation

$$\sum_{i=1}^{3} p_i \dot{q}_i = \sum_{i=1}^{3} v_i \dot{r}_i = \sum_{i=1}^{3} v_i^2 \qquad \begin{array}{l} p_i = v_i \\ q_i = r_i \end{array}$$

Let $Q_i = u_i$, $P_i = 4u^2 \dot{u}_i$ Levi-Civita variables

$$\sum_{i=1}^{4} P_i \dot{Q}_i = 4u^2 \sum_{i=1}^{4} \dot{u}_i^2 = \sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{3} p_i \dot{q}_i$$

since $\sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{4} 4u^2 \dot{u}_i^2$ see before

2) Time transformation $dt = g(Q,t)\, ds$

$$\delta S = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{3} p_i \dot{q}_i - H(p,q) \right) dt = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{4} P_i \dot{Q}_i - H(P,Q) \right) dt$$

$$= \delta \int_{S_1}^{S_2} \left(\frac{1}{g} \sum_{i=1}^{4} P_i Q_i' - H(P,Q) \right) g\, ds$$

$$\boxed{Q_i' = \frac{\partial Q_i}{\partial s} = \frac{\partial Q_i}{\partial t} \frac{\partial t}{\partial s} = \dot{Q}_i \cdot g \,;\; \Gamma = g\left(H - \frac{h}{\mu}\right)} = \delta \int_{S_1}^{S_2} \left(\sum_{i=1}^{4} P_i Q_i' - \Gamma \right) ds$$

$p_i, q_i, H(p_i, q_i, t), t$ satisfies Hamilton's eq.

$$\Longleftrightarrow$$

$P_i, Q_i, \Gamma(P_i, Q_i, s), s$ satisfies Hamilton's eq.

Γ: Poincaré transform of Hamiltonian

$$\Gamma = g\left(H - \frac{h}{\mu}\right) \qquad dt = g \cdot ds$$

Our example: $g = r = u^2 = \sum_{i=1}^{4} u_i^2$

$$H = \frac{1}{2}\sum_{i=1}^{3} v_i^2 - \frac{Gm_1m_2}{\mu r} = 2u^2 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu u^2}$$

$$\Gamma = r\cdot\left(H - \frac{h}{\mu}\right) = 2u^4 \sum_{i=1}^{4} \dot{u}_i^2 - \frac{Gm_1m_2}{\mu} - \frac{h}{\mu}u^2$$

$$= \sum_{i=1}^{4}\left(\frac{P_i^2}{8} - \frac{h}{\mu}Q_i^2\right) - \frac{Gm_1m_2}{\mu}$$

Canonical Equations:

$$Q_i' = \frac{\partial \Gamma}{\partial P_i} = \frac{P_i}{4} \Longleftrightarrow u_i' = u^2 \dot{u}_i = r\dot{u}_i \quad \text{O.K.}$$

$$P_i' = -\frac{\partial \Gamma}{\partial Q_i} = 2\frac{h}{\mu}Q_i \Longleftrightarrow \left(4u^2\dot{u}_i\right)' = 4u_i'' = 2\frac{h}{\mu}u_i$$

$$\boxed{u_i'' - \frac{h}{2\mu}u_i = 0 \quad \Big| \quad u_i'' + \frac{|h|}{2\mu}u_i = 0 \quad (h<0)}$$