The issue about time symmetry

1) Excursion: "leap frog" — simplest form $O(\Delta t^2)$
"natural" time symmetry, Hamilton

Hamiltonian relative two-body motion, gravitational:

$$H = \frac{1}{2}\mu v^2 - \frac{GMm}{r}$$

; $v = |\vec{v}_1 - \vec{v}_2|$ relative vel.
$r = |\vec{x}_1 - \vec{x}_2|$ " distance
m, M two masses
$\mu = mM/(m+M)$ reduced mass

Separable:

$$H = \underbrace{H_1(v)}_{\text{kin. En.}} + \underbrace{H_2(r)}_{\text{pot. Energy}}$$

$p = \mu v$; $H = \frac{p^2}{2\mu} - \frac{GMm}{r}$

Conservative system

$$H = H_1 + H_2 = \frac{1}{2}\mu v^2 - \frac{GMm}{r}$$

canonical equations $\dot{r} = \frac{\partial H}{\partial v} \overset{\text{Sep.}}{=} \frac{\partial H_1}{\partial v} = \mu \cdot v = p$

$$\dot{v} = -\frac{\partial H}{\partial r} = -\frac{\partial H_2}{\partial r} = -\frac{GMm}{r^2}$$

Newton's Law

„Leap Frog Schem"

Time Step: t_0, t_1 ; $\Delta t = t_1 - t_0$

$$\frac{v_1 - v_0}{\Delta t} = -\frac{\partial H_2}{\partial r} = -\frac{GMm}{r_{1/2}}$$

t_0 $r_{1/2}$ t_1

v_0 v_1

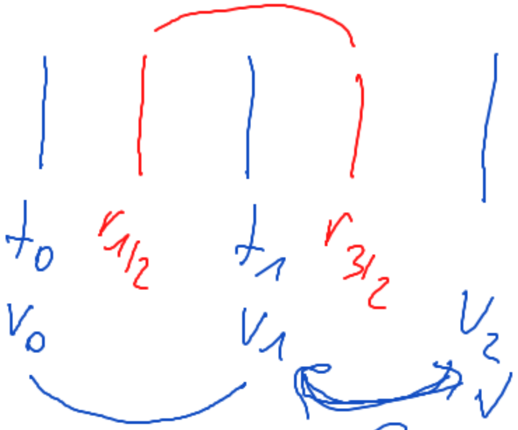
$$v_1 = v_0 - \Delta t \cdot \frac{GMm}{r_{1/2}} + O(\Delta t^3)$$

because of centering $r_{1/2}$

$$v_1 = v_0 - \Delta t \frac{GMm}{r_{1/2}} + \mathcal{O}(\Delta t^3)$$

centering

$$r_{3/2} = r_{1/2} + \Delta t \cdot v_1 + \mathcal{O}(\Delta t^3)$$



Leap Frog

$$v_1 = v_0 - \Delta t \left.\frac{\partial H_2}{\partial r}\right|_{r=r_{1/2}}$$

$$r_{3/2} = r_{1/2} + \Delta t \cdot \left.\frac{\partial H_1}{\partial v}\right|_{v=v_1}$$

direct discretization of Hamiltonian equations! Can only do leap frog because H is separable!

Solutions of conservative Hamiltonian systems are time-symmetric!
Numerical Leap Frog is exactly " "

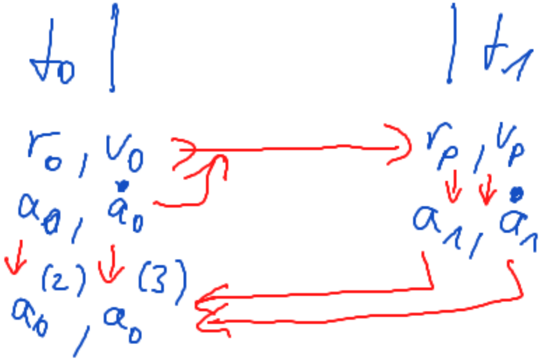
$\Delta t' = -\Delta t$

- Time Symmetry is nice feature of Leapfrog
- Numerical solution, using leap frog, is an exact solution of an approximate Hamiltonian $H = \underline{\underline{H_0}} + \underline{\underline{\Delta t}} \cdot \underline{\underline{H_1}} + \underline{\underline{\dots}}$

  $t_0 \to t_1$; Δt constant timestep
- Hermite Scheme not time symmetric

  has a global energy error

  keep very small!



*-cluster simulations tolerated! trust Q + Tremaine 1992

- Two possible ways out:

a) higher order „leap frog" or similar („symplectic" integrator)
BIFROST Rantala, Naab et al. 2020, ...

b) make Hermite scheme symmetric!
Kokubo + Makino (2004)

Classical Hermite corrector:

$$\vec{r}_c(t_1) = \vec{r}_p(t_1) + \frac{1}{24}\Delta t^4 \vec{a}_0^{(2)} + \cancel{\frac{1}{120}\Delta t^5 \vec{a}_0^{(3)}} + O(\Delta t^6) \quad + O(\Delta t^5)$$

$$\vec{v}_c(t_1) = \vec{v}_p(t_1) + \frac{1}{6}\Delta t^3 \vec{a}_0^{(2)} + \frac{1}{24}\Delta t^4 a_0^{(3)} + O(\Delta t^5)$$

Hermite Formula $a_0^{(2)}, a_0^{(3)}$

$\Updownarrow$

Strictly Time Symmetry

$$\vec{r}_c(t_1) = \vec{r}_0(t) + \frac{1}{2}\Delta t(\vec{v}_c + \vec{v}_0) - \frac{1}{12}\Delta t^2(\vec{a}_1 - \vec{a}_0) + O(\Delta t^5)$$

$$\vec{v}_c(t_1) = \vec{v}_0(t) + \frac{1}{2}\Delta t(\vec{a}_1 + \vec{a}_0) - \frac{1}{12}\Delta t^2(\dot{\vec{a}}_1 - \dot{\vec{a}}_0) + O(\Delta t^5)$$

t_0: r_0, v_0; $a_0, \dot{a}_0$

t_1: $a_1, \dot{a}_1$; r_c, v_c

$\Delta t' = -\Delta t$

Exchange 0, 1

r_0, r_c ; v_0, v_c

- Was proposed for high accurate orbit integrations suitable for planetary systems, particles in protoplanetary disk.
- Caveat: Whenever time step varies $\leftrightarrow$ symmetry is broken

$$t_0 \longrightarrow t_1$$

$$\Delta t_{\rightarrow} = \Delta t\left(a_0, \dot{a}_0, a_0^{(2)}, a_0^{(3)}\right)$$

$$t_0 \longleftarrow t_1$$

$$\Delta t_{\leftarrow} = \Delta t\left(a_1, \dot{a}_1, a_1^{(2)}, a_1^{(3)}\right)$$

Currently „time-symmetric" Hermite is not used: due to problems with variable time steps and Ahmad-Cohen neighbour scheme