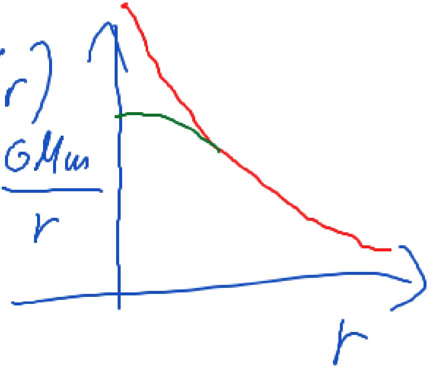
Regularization: Decision making → postpone

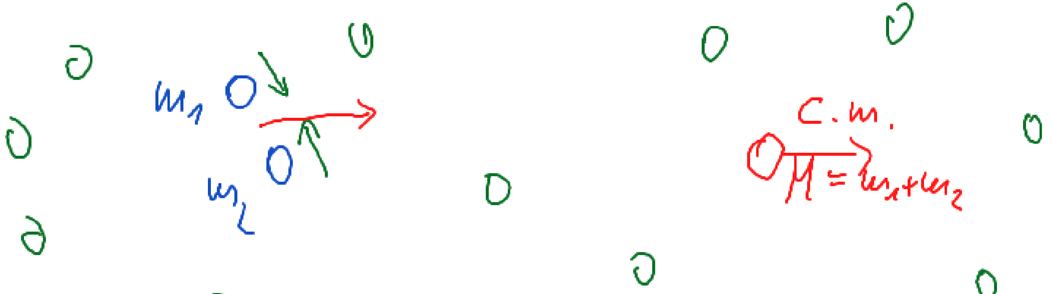
Since von Hoerner very close encounters delay or even stop large N-body simulation

2 Ways out: 1) Softening of grav. 2-body potential (ε)

2) Regularization
Aarseth, Mikkola (198- 199-)

$$-V(r) = +\frac{GMm}{r}$$

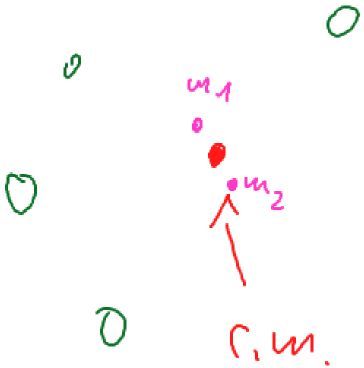




center-of-mass particle

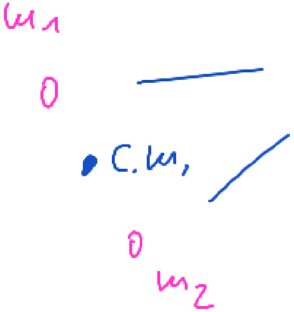
center-of-mass c.m. in Hermite integrators,

internal motion of 2-b in regularized coord.



- Regularized two-body integration transforms the singular 2-body problem into a regular harmonic oscillator
- here: show isolated 2-body problem is regularized, unusual way
- what about interactions?

- „perturbing" forces from other particles on KS pair



- resolving KS pair for nearby particles

- In Hermite: can resolve KS pair for near particles
- In KS (Kustaanheimo + Stiefel 1965) 2-body regularization we take into account perturbing forces of other nearby particles (perturber list ~ neighbor list)
- Third particle approaches KS pair; we can allow the „perturbing" force by m_3 to grow, to something ~50% of two-body force

„perturber" $\longleftrightarrow$ external force

- Post-Newtonian relativistic force in KS pair as external force

Start with Regularization in 1D

2b Hamiltonian: (relative motion)

$$H = \frac{p^2}{2\mu} - \frac{GMm}{r} = E_0 = \text{const.}$$

(isolated system)

$M, m \Leftrightarrow m_1, m_2$ $\qquad \mu = mM/(m+M)$

Canonical Variables: r, p $\qquad$ r: Separation

Can. Transformation: $r = u^2$, P $\qquad$ $v = \dot{r}$: rel. velocity

$\qquad$ $p = \mu \cdot v$ linear momentum

$$p\dot{r} = P\dot{u}$$

u, P canonically conjugate

$$\dot{r} = 2u\dot{u}$$

$$\mu\dot{r}^2 = P\dot{u} = \mu \cdot 4u^2\dot{u}^2$$

$$P = 4\mu u^2\dot{u}$$

$$r = u^2 \quad ; \quad P = 4\mu u^2 \dot{u} = 2u \cdot p$$

1. Canonical Transformation; canonical eqs. of motion for the transformed Hamiltonian:

$$H = \frac{p^2}{8\mu u^2} - \frac{GMm}{u^2} = \text{const.} = E_0$$

$$H = \frac{p^2}{2\mu} - \frac{GMm}{r}$$

$$\boxed{\dot{P} = -\frac{\partial H}{\partial u} \quad ; \quad \dot{u} = \frac{\partial H}{\partial p}}$$

Still H is singular now in u^2 (instead of r)

2. Time transformation:

$$\underset{\text{phys. time}}{dt} = r\, ds = u^2\, ds$$

same function of u, E, r, P

fictitious time transformed

"Stretching of time" if r becomes small

$$\dot{u} = \frac{du}{dt} = \frac{1}{r}\frac{du}{ds} = \frac{1}{r}u' \qquad \Big| \cdot r \qquad u^2$$

$$u^2 \dot{u} = u' = \frac{1}{4}\frac{P}{\mu}$$

3) Poincaré Transformation of Hamiltonian

$$\Gamma_{\text{Prel.}} = r \cdot H(P, u)$$

Poincaré-T.

any time transformation function

New canonical eqs. using $\Gamma_{\text{Prel.}}$: $H = E_0 = \text{const}$

Γ_{Prel} is <u>not</u> constant!

It is allowed to add / subtract a constant from H!

$$H_{new} = H(P, u) - E_0 = 0$$

$$\Gamma = r \cdot H_{new}(P, u) = 0 \quad \text{const.!}$$

$$\Gamma = r \cdot \left(H(P,u) - E_0 \right) = 0 \quad \text{const.}$$

$$= u^2 \left(\frac{p^2}{8\mu u^2} - \frac{GMm}{u^2} - E_0 \right)$$

$$\boxed{\Gamma = \underbrace{\frac{p^2}{8\mu} - E_0 u^2}_{\text{harmonic}} - GMm}$$

$E_0 < 0$

Γ is regular

Γ is Poincaré Transform.

Canonical Equations: $p' = -\dfrac{\partial \Gamma}{\partial u}$; $u' = \dfrac{\partial \Gamma}{\partial p}$

are valid for Poincaré-transformed Hamiltonian

canon. + time transform. $\Rightarrow$ Principle of least ~~eq~~ action!

$$p' = -\frac{\partial \Gamma}{\partial u} = +2E_0 u \; ; \qquad u' = \frac{\partial \Gamma}{\partial P} = \frac{p}{4\mu}$$

$$\Gamma = \frac{p^2}{8\mu} - E_0 u^2 - GMm$$

$$u'' = \frac{p'}{4\mu} = \frac{2E_0 u}{4\mu}$$

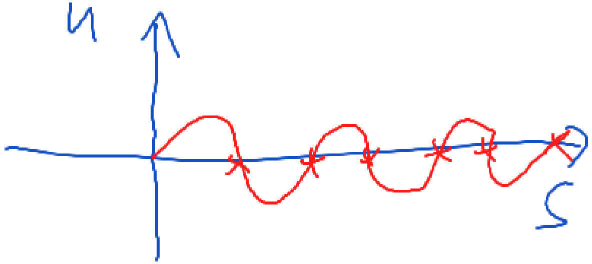
$$\boxed{u'' - \frac{E_0}{2\mu} u = 0}$$

harmonic oscillator equation

$$\omega^2 = \frac{|E_0|}{2\mu} \qquad E_0 < 0$$

numerically solve because of perturbations

solution is regular for $u=0 \Rightarrow r=0$



Singularity of two-body grav. problem is removable

- Mathematically 1D regularization allows steady solutions across $r = u^2 = 0$
- Practically in N-body code we do NOT use this property; we use KS regularization for perturbed pairs ONLY.
- If a KS pair becomes unperturbed
  → switch to analytical solution of Keplerian orbit

$$\boxed{u'' + \frac{|E_0|}{2\mu} u = \text{perturbing forces} \qquad \text{numerically solve KS integration}}$$

- What about to 2D?

$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\dot{\vec{r}} = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

- $\vec{r} = \vec{u}^2$?? ~~scalar product~~?

"conformal" mapping $r \Leftrightarrow u$

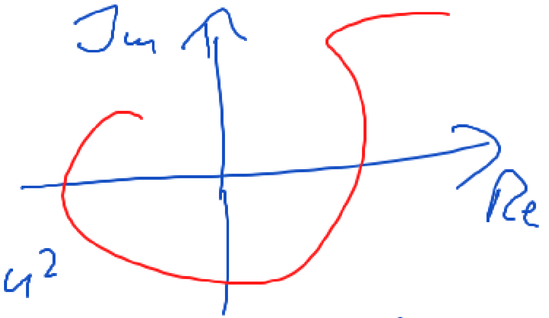
$\vec{r} = 0 \Longleftrightarrow \vec{u} = 0$

Only way: complex number algebra,

complex number $u = u_1 + i u_2$

$u^2 = u_1^2 - u_2^2 + i \cdot (2 u_1 u_2)$

$$r = r_1 + i r_2 \; ;$$



$$r^2 = u_1^2 - u_2^2 + i\,(2u_1u_2) = u \cdot u = u^2$$

Complex number multiplication

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \; ; \; u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$r = u^2 = L(u) \cdot u = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{pmatrix}$$

matrix representation of complex number algebra

$L(u)$ = Levi-Civita matrix

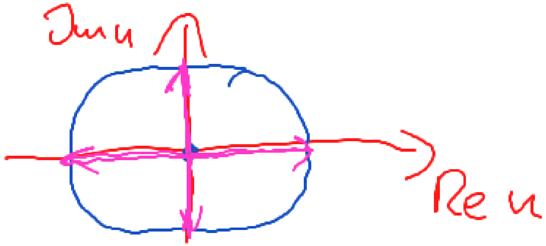
- $r = u^2 = L(u)\,u \quad ; \quad \dot{u} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}$

  $\dot{r} = 2u\dot{u} = 2\,L(u)\,\dot{u}$

- $H = \frac{p^2}{2\mu} - \frac{GMm}{r} = E_0 \quad ;$ time transf. $dt = u^2 ds$

- $\Gamma = u^2 \left(H(P,u) - E_0 \right) = \frac{P^2}{8\mu} - E_0 u^2 - GMm$

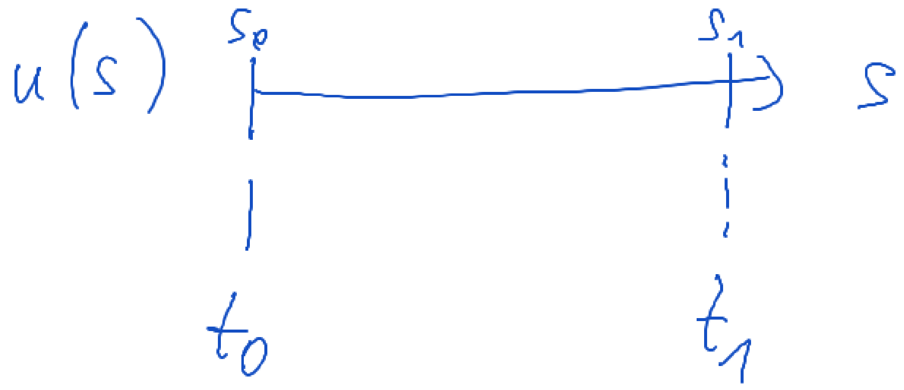
- harmonic oscillator: $u'' - \frac{E_0}{2\mu} u = 0$ [pert. force]



$r = u^2 \iff \dots \quad u \longrightarrow r$

$t \qquad\qquad\qquad s$

- Solution of numerical KS integration



$$dt = u^2 \, ds$$

- Relation between t and s is non-linear and also done with Taylor series
- Two Body Problem is planar if isolated

- Because we have 3D perturbations on our binary we need 3body regularization!

- $\vec{r} = \vec{u}^2$ $\quad \vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$; $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\quad \vec{r} = 0 \Leftrightarrow \vec{u} = 0$

  Does not exist in 3D!

- Next larger dimension of a field (Körper) is 4D:

  Quaternion algebra

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \end{pmatrix} \qquad r = \begin{pmatrix} r_1 \\ \vdots \\ r_4 \end{pmatrix} \qquad r = u^2 \; ; \; u = \begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix} \; ; \; \underline{r = u^2 = L_4(u)\, u}$$

- 1D: canonical transf. $r, p \to u, P$; time transformation; Γ Poincaré transform of H; harmonic oscillator eq. of motion for $u = u(s)$
- 2D: generalization complex numbers, ok
  $r = u^2 = u \cdot u$
- 3D: impossible, no 3D field
- 4D: quaternions
- Problem

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \end{pmatrix}; \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix} \xrightarrow[\text{oscillators}]{\text{harmonic}} u(s) \xrightarrow[\text{back}]{\text{transf.}} \begin{pmatrix} r_1 \\ \vdots \\ r_4 \end{pmatrix}$$

Quaternion space $\mathbb{Q}$: $u = \underline{u_1 + iu_2} + \underline{ju_3 + ku_4}$

Excursion: multiplication tables of objects of $\mathbb{C}, \mathbb{Q}$ enough to define multiplications of basis elements

$\mathbb{C}$	1	i
1	1	i
i	i	-1

$$\Longrightarrow \quad u \cdot u = \underline{L(u)u} \; ; \; L(u) = \underline{\begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}}$$

$$(u_1 + iu_2)(u_1 + iu_2) = \ldots$$

Q	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

$$(u_1 + iu_2 + ju_3 + ku_4)\cdot$$
$$(\quad - \ldots \quad)$$

$$u^2 = L(u)u$$

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & u_4 & -u_3 \\ u_3 & -u_4 & u_1 & u_2 \\ u_4 & u_3 & -u_2 & u_1 \end{pmatrix}$$

$$r = u^2 = \Rightarrow L(u)u = \begin{pmatrix} \underline{u_1^2 - u_2^2}\,\underline{- u_3^2 - u_4^2} \\ \underline{2u_1u_2} \\ \underline{2u_1u_3} \\ \underline{2u_1u_4} \end{pmatrix}$$

Kustaanheimo + Stiefel 1965:

Little Manipulation in Quaternion multiplication

	1	i	j	k
1	1	i	j	−k
i	i	−1	k	j
j	j	−k	−1	−i
k	k	j	−i	+1

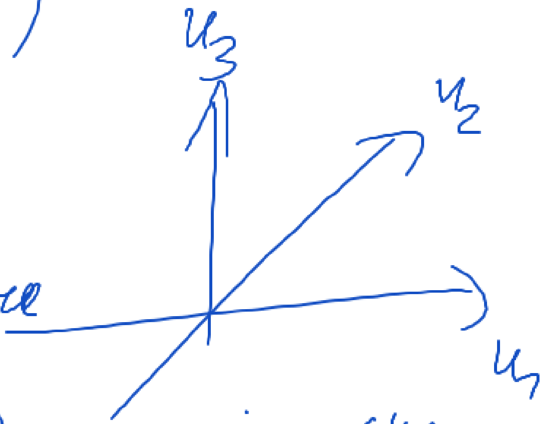
strange

$$L_{KS}(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & +u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}$$

$$u^2 = L_{KS}(u) \cdot u = \begin{pmatrix} u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ 2u_1u_2 - 2u_3u_4 \\ 2u_1u_3 + 2u_2u_4 \\ 0 \end{pmatrix} \quad \checkmark$$

- We found a conformal transformation: $r = u^2 = L(u)u$
  $r, u \in \mathbb{Q}_{KS}$ $\quad u^2 = L(u)u = \begin{pmatrix} x \\ x \\ x \\ 0 \end{pmatrix}$ 3D space

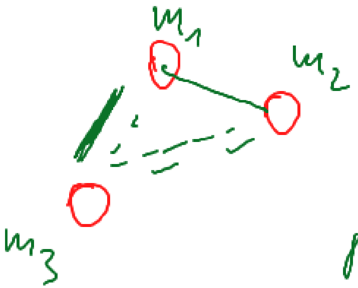
- Kustaanheimo + Stiefel (1965)
  Reduced, specially $\mathbb{Q}$ algebra



diff. $u'' - \frac{E_0}{2\mu} u' = 0$, pert. force

Solving unusually in 4D $u \in \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \Rightarrow u^2 = \begin{pmatrix} x \\ x \\ x \\ 0 \end{pmatrix}$

- 3D regularization for 2 bodies using $u \in \mathbb{H}$ 4D
- What about more than 2 bodies?
  3b, 4b, few-body regularizations, chain, alg. regularization
- Don't believe that there is regularization for more than two bodies — it is kind of cheating!
- All higher (claimed) regular. (more than 2 bodies) are combinations of 2-b-regularizations!



Aarseth + Zare 1974 3b regul.

$R_1 = |r_2 - r_1|$; $R_2 = |r_3 - r_1|$

$R_1 = u_1^2$ $\qquad$ $R_2 = u_2^2$

Why is 3b regularization generally impossible?

$$\vec{R} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \\ r_4 \\ r_5 \\ r_6 \\ 0 \end{pmatrix} = u^2 = L(u)\,u$$

$\vec{R} \in 8D$

① octave space

$L(u)$: 8x8 matrix

manipulation in multiplication table

$u^2 = (\cancel{6 \text{ numbers}, 0}, 0)$

Found: $u^2 = (7 \text{ numbers}, 0)$

Or: $u^2 = (5 \text{ numbers}, 0, 0, 0)$

m_1 0 R_2 , R_1 0 m_2 0 m_3

$R_1, R_2 \in \mathbb{R}^3$

$R = u^2$

impossible for 3D

" for 6D

Octaves 8D

Cayley's Numbers

prime subgroups

$Q_{4D} \Leftrightarrow (3,1)$

$O_{8D} \Leftrightarrow (7,1)\ (5,3)$

- Fortunately way out in astrophysics:

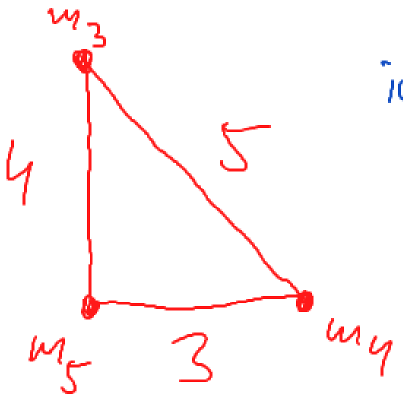
Sundmann's Theorems:

In 3-body system the space of initial conditions I

Let I_{coll} is space of initial cond's which lead to a mathematical 3-body collision; $\dim I_{coll} = 0$

- In astrophysical N-body systems there is always a sequence of close two-body encounters; even in compact subsystems 3b or higher collisions improbable!

- Illustration: Famous 3-body problem
Burrau's problem Szebehely (1967)



initial velocity zero

$m_3 = 3$; $m_5 = 5$; $m_4 = 4$

$\longrightarrow$ sequence of extremely close two-body encounters

- List of „so-called" higher regularizations:

1) Aarseth+Zare 1974 $\cancel{R_1}\,\cancel{R_2}$ R_3 $\quad R_1 = u_1^2\,;\ R_2 = u_2^2$

Time transformation: $dt = R_1 R_2\, ds$

- Poincaré-Transform of Γ for 3b-AZ regularization:

$$\Gamma = R_1 R_2 \left(H_{3b} - E_0 \right) = u_1^2 u_2^2 \left(H_{3b}\left(u_1, u_2, P_1, P_2\right) - E_0 \right)$$

$$= \propto \; P_1^2, P_2^2 \; ; \quad u_1, u_1^2 \; + \ldots \frac{G m_1 m_2\, u_1^2 u_2^2}{|\vec{R}_1 - \vec{R}_2|}$$

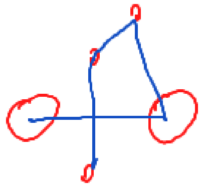
Γ is regular w.r.to R_1, R_2

singular w.r.to $|\vec{R}_1 - \vec{R}_2|$ R_3

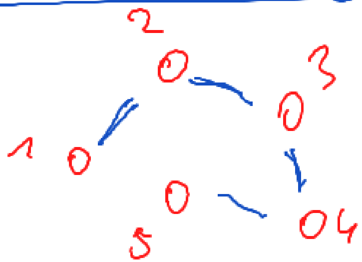
- Heggie's global regularization for N-bodies

$$dt = \prod_{\text{all pairs}} R_{ij} \; ds$$

- Chain regularization, few body system in chain



MSTAR



$\dfrac{\text{distances}}{\text{forces}}$

Separate from large outside N-body system

5 bodies ⟷ 4 pairwise distance (Mikkola + Aarseth)

„perturbing" force from other bodies

Nbody6++GPU: using 2-body KS, chain for more than 2 bodies

- NBODY7: algorithmic regularization | Preto + Tremaine 99
Mikkola + Tanikawa 1999, Mikkola + Merritt 2008

$$H = H_1(p) + H_2(r) = T(p) + U(r)$$

Time transformation: $dt = \frac{1}{|U|} ds$ $\quad U = -\sum_{\substack{\text{pairs} \\ r_{ij}}} \frac{G m_i m_j}{r_{ij}}$

If e.g. r_{12} becomes small $\to 0 \Rightarrow |U| \to \frac{1}{r_{12}}$

$$\Gamma = |U|^{-1} \left(T(p) + U(r) - E_0 \right) = 0$$

$$\Gamma = |U|^{-1}\left(T(p) + U(r) - E \right)$$

Strange: This Γ is not separable anymore!

$$\Lambda = \log(1+\Gamma)$$

$$\Gamma = \frac{T(p)}{|U|} - 1 - \frac{E}{|U|}, \quad 1+\Gamma = \frac{T(p)-E}{|U|}$$

$$\log(1+\Lambda) = \log\left(T(p)-E\right) - \log|U|$$

Separable! logarithmic Hamiltonian

Canonical Equations for $\Lambda = \log(1+\Gamma)$

NB30O17

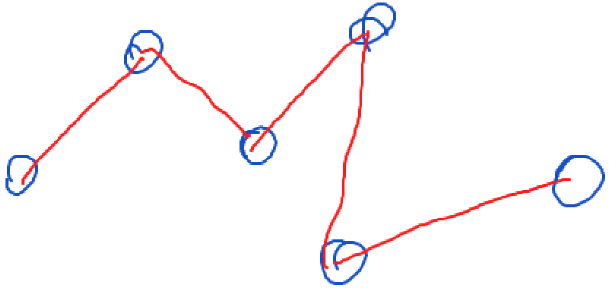
$$p' = -\frac{\partial\Lambda}{\partial r} = -\frac{1}{1+\Gamma}\frac{\partial\Gamma}{\partial r}$$

$$r' = \frac{\partial\Lambda}{\partial p} = \frac{1}{1+\Gamma}\frac{\partial\Gamma}{\partial p}$$

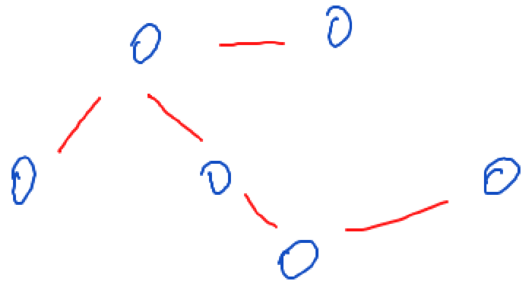
$$t' = \frac{\partial\Lambda}{\partial p_0} = \frac{1}{1+\Gamma}\frac{\partial\Gamma}{\partial p_0}$$

- There is no KS transformation
- We use canonical eqs. for Λ with classical p, r
- time transformation $dt = |u|^{-1} ds$

Algorithmic Regularization



classical + AR chain



MSTAR, BIFROST / Rautala, Spiegel / Naab

minimum spanning tree