Statistical Methods

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Exercise 5 for **August 9**, 2024, 18:00

Samples with Monte Carlo, Covariance

5.1 Random samples with a Monte Carlo Markov chain

Let's assume we want to generate random numbers from the distribution

$$p(x) \propto \exp\left(-\left[x + 2\cos^2(x)\right]^2\right).$$

This simple case could also be done with the rejection method, but here we want to adopt a different approach, namely the use of a stochastic process constructed with the Metropolis algorithm. Note the sign of proportionality: $p(x)$ as given above is not normalized! Have a look at $p(x)$ first, to see over which region the function is noticeably different from zero.

a: Start with some random guess x_0 for which $p(x)$ is not zero!

b: Make a proposal for x_i' in your chain by adding a random number drawn uniformly from the interval $[-1, 1]$ to x_{i-1}!

c: Accept the proposal with probability

$$r = \min\left(1, \frac{p(x_i')}{p(x_{i-1})}\right), \tag{1}$$

i.e. in the case of acceptance make it the entry x_i in your Monte Carlo chain. Otherwise, adopt the unmodified x_{i-1} as your element x_i. Then proceed with the next element $i+1$!

d: Produce a chain with $N = 10^6$ elements, and make a histogram with bin size $\Delta x = 0.02$ of the entries in order to verify that they correctly sample the overplotted shape of $p(x)$. How many *different* points are in your chain?

5.2 Confirming some numbers (analytical)

During the lecture we came across the question what $\mathrm{Cov}\left[x, x^2\right]$ might be. After practicing calculating covariances analytically, it should not be a big deal for you to confirm the following:

a: if $x \sim N(0,1)$: $\mathrm{Cov}\left[x, x^2\right] = 0$, $\mathrm{Cor}\left[x, x^2\right] = 0$

b: if $x \sim U(0,1)$: $\mathrm{Cov}\left[x, x^2\right] = \frac{1}{12}$, $\mathrm{Cor}\left[x, x^2\right] = \frac{1}{4}\sqrt{15}$

c: in case a) x and x^2 are uncorrelated. Are they also independent?

The "$\sim$" means "distributed as".